Quasi-stationary distributions for Markov processes and its applications

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Introduction

We study the problem of the existence and uniqueness of quasi-stationary distributions (QSD). The study of quasi-stationary distributions for branching processes began with the work of Russian Mathematician A. M. Yaglom. In [i], P.A.Ferrari et al. study the problem for general continuous Markov Processes and E.A.van Doorn consider the problem for birth and death processes. In particular in [ii] a necessary and sufficient condition for the existence and uniqueness is obtained. By applying this result, we check the existence and uniqueness of quasi-stationary distribution for three typical models, birth and death processes with constant rates, linear birth and death processes and logistic birth and death processes. We consider the application of quasi-stationary distribution on field of demography or biology.

The general theory for QSD

Notations and Assumptions Let \( \{X_t : t \leq 0\} \) be a continuous time Markov process on a state space \( E \) of form \( 0 \cup \{1, 2, \ldots\} \), where 0 is an absorbing state and \( E^* := \{1, 2, \ldots\} \) is an irreducible transient class. Let \( Q \) be the corresponding transition rate matrix whose component \( q_{ij} \) represents the rate of jumping from \( i \) to \( j \). We assume that \( Q \) is conservative and honest. Let \( T_0 \) be the first hitting time at 0, \( T_0 = \inf\{t \geq 0 : X_t = 0\} \). We further assume that the expectation of \( T_0 \) is finite for any \( i \in E^* \).

Definitions

1) \( \alpha \in P(E^*) : \) quasi-stationary distribution (QSD)
\[ \alpha(A) = \mathbb{P}_\alpha(X_t \in A | T_0 > t), \forall t \geq 0, A \subset E^* \]
2) \( \alpha \in P(E^*) : \) quasi-limiting distribution (QLD)
\[ \exists \nu \in P(E^*) \ s.t. \]
\[ \alpha(A) = \lim_{t \to \infty} \mathbb{P}_\nu(X_t \in A | T_0 > t), \ A \subset E^* \]
3) \( \alpha \in P(E^*) : \) Yaglom limit
\[ \alpha(A) = \lim_{t \to \infty} \mathbb{P}_\nu(X_t \in A | T_0 > t), \forall x \in E^*, A \subset E^* \]
QSD \iff QLD \iff Yaglom limit.

The existence of QSD

Under the above assumptions and the following condition:
\[ \lim_{i \to \infty} \mathbb{P}_i(T_0 < t) = 0 \quad \forall t > 0, \ i \in E^* \]
we see from [i] that the existence of quasi-stationary distribution is equivalent to
\[ \mathbb{E}_x [e^{\lambda T_0}] < \infty, \]
for some \( \lambda > 0 \) and for some \( i \in E^* \) (and hence for all \( i \)).

In case of birth and death processes

Notations and Assumptions We consider a birth and death process on \( E \) with birth coefficients \( \lambda_i \) and death coefficients \( \mu_i, \) for \( i \in E \). We assume that \( \lambda_0 = \mu_0 = 0, \lambda_i > 0, \mu_i > 0, \ i \geq 1 \). We further assume that the eventual absorption at 0 is certain. This condition is equivalent to
\[ \sum_{i=1}^{\infty} \frac{1}{\lambda_i \mu_i} = \infty \]
where \( \pi_1 = 1; \pi_i = \frac{\lambda_1 \lambda_2 \cdots \lambda_{i-1}}{\mu_2 \mu_3 \cdots \mu_i}, \ i = 2, 3, \ldots. \]
In case of birth and death processes, the transition probabilities is given by
\[ P_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) d\psi(x), \]
where \( \psi \) is the unique positive probability measure on \([0, \infty)\). Let \( \xi \) be the infimum of support of \( \psi \). Then \( \xi \) plays a crucial role on the existence of quasi-stationary distribution together with the following sum:
\[ S = \sum_{n=1}^{\infty} \frac{1}{\lambda_n \mu_n} \sum_{i=n+1}^{\infty} \pi_i \]
The existence of QSD

Theorem (iii) A necessary and sufficient condition for the existence of QSD is as follows;

1) If the sum \( S \) converges, then \( \xi > 0 \) and there is precisely one QSD.
2) If the sum \( S \) diverges and \( \xi = 0 \), then there is a one-parameter family of QSDs.
3) If the sum \( S \) diverges and \( \xi = 0 \), then there is no QSD.

Examples

1) birth and death processes with constant rates; If \( \lambda_i = \lambda, \ \mu_i = \mu, \ i \geq 0 \), then the Yaglom limit is \( \alpha_j = j(1-\beta)^{j-1}, \) for \( j = 1, 2, \ldots \).
2) linear birth and death processes; If \( \lambda_i = i\lambda, \ \mu_i = i\mu, \ i \geq 0 \), then the Yaglom limit is \( \alpha_j = (1-\lambda/\mu)(\lambda/\mu)^{j-1}, \)
3) logistic birth and death processes; If \( \lambda_i = \lambda i, \ \mu_i = \mu_i + ci(i-1), \ i \geq 0 \), the infinite is a entrance boundary in Feller sense. This implies that the sum \( S \) converges. So, there is a unique QSD.

The application of QSD

In demography or biology, the extinction rate of \( X \) starting form \( \mu \) at time \( t \geq 0 \) is given by
\[ \gamma_\mu(t) = \frac{\partial}{\partial t} \mathbb{P}_\mu(T_0 > t). \]
If \( \alpha \) is a QLD for \( X \) started from a probability measure \( \mu \) on \( E^* \), then
\[ \lim_{t \to \infty} \gamma_\mu(t) = \gamma_\alpha(0). \]
That is, the existence of a QLD for \( X \) with initial distribution \( \mu \) implies the existence of a long term mortality plateau.

References