**Formal Group Law**

Let $R$ be a commutative ring with the identity. Let $X = (x_1, x_2, \ldots, x_n)$ and $R[X]$ be the ring of formal power series with coefficients in $R$.

$F(X, Y) = (F_0(X, Y), \ldots, F_n(X, Y))$, $F(X, Y) \subset R[X, Y]$ is a formal group law over $R$ if

1. $F(Z, 0) = F(0, Z) = Z$
2. $F(F(X, Y), Z) = F(X, F(Y, Z))$

**Examples of (a one-parameter) Formal Group Law**

The additive formal group law:

$G_a(x, y) = x + y$

The multiplication formal group law:

$G_m(x, y) = (x + 1)(y + 1) - 1$

For $f(x) \in R[X]$ such that $f(x) = ax + \cdots$ ($a$ is a unit in $R$), $F(X, Y) = f^{-1}(f(x) + f(y)) \in R[X, Y]$ is a formal group law over $R$. $f(x)$ is called the logarithm of $F(X, Y)$.

**Elliptic Curve**

An elliptic curve $E$ over a field $K$ is a nonsingular projective cubic curve, defined over $K$, with a $K$-rational point. $P = (x_1, \ldots, x_n)$ is a $K$-rational point if $x_i \in K$. We write $P \in E(K)$.

An elliptic curve over $\mathbb{Q}$ can be expressed in Weierstrass form $\tilde{E}$:

$\tilde{E}: y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \in \mathbb{Q}$

and by taking the minimal model of $\tilde{E}$, $a_i$ will be in $\mathbb{Z}$. In general, a line through $P$ meets $E$ at 3 points. By using this fact, we can define an addition on an elliptic curve and the points on $E$ form a group. This addition can be expressed as a formal group law (the method I).

**Honda's Theorem (1968)**

Let $E$ be an elliptic curve over $\mathbb{Q}$, $E_p$ be the Weierstrass minimal model of $E$, $E_p \equiv E_w \mod p$ for a prime number $p$, and $E(x, y)$ be the formal group law(!) given by the group law of $E$. For each prime number $p$, $L_p$ is defined as:

$L_p = \begin{cases} (1 - a_pp^{-2} + p^{-2}b), &\text{if } E_p \text{ is of genus one;} \\ (1 - e_pp^{-2})^{-1}, &\text{if } E_p \text{ has an ordinary double point; } \\ 1, &\text{if } E_p \text{ has a cuspidal.} \end{cases}$

where $a_p = p + 1 - \#E(\mathbb{F}_p)$, and $e_p = 1$ if the tangents at the double point are rational over $\mathbb{F}_p$ and $e_p = -1$ if not. We define

$L(x) = \prod_{n \geq 1} L_p(s)^{n} = \prod_{n \geq 1} \prod_{p \mid n} L_p(s)^{n}$

$L(x, y) = l^{-1}(l(x) + l(y))$.

Let $S$ be any set of prime numbers which do not contain 2 (resp. 3) if $E_2$ (resp. $E_3$) has genus one with $a_2 = \pm 2$ (resp. $a_3 = \pm 3$), and $Z_S = \prod_{p \in S} (\mathbb{Z} \cap \mathbb{Q})$. Then

$L(x, y)$ is a formal group law (the method II) over $\mathbb{Z}_S$.

$L(x, y) \sim E(x, y)$ over $\mathbb{Z}_S$.

*The restriction about $S$ is removed by his paper in 1970. In this theorem, there are two ways of the construction of a formal group law, (I) and (II).*

**Formal Group**

For a ring $R$, let $\text{NilR-alg}_R$ be the category of nil-$R$-algebras, i.e. the ideals of nilpotent elements of $R$-algebras. Formal affine $R$-space over $R$ is the functor

$\Pi_R : \text{NilR-alg}_R \to \text{Sets}$

which assigns to a nil-$R$-algebra $A$ the set $A^n$ and to the morphism $f$ the map $f^n$. An $n$-dimensional formal group over $H$ is a functor $G : \text{NilR-alg}_R \to \text{Sets}$ such that $F_{\text{tr}} \circ G \cong A^n$, where $F_{\text{tr}}$ is the forgetful functor. (e.g.) If $F(X, Y)$ is a formal group law over a commutative ring $R$, a commutative nil-$R$-algebra $A$ inherits a group structure from $F$ by $a + b = \text{F}(a, b)$ for $a, b \in A$, where $\text{F}(a, b)$ is a convergent sum. So this induce a formal group.

**Artin-Mazur Functor**

Let $R$ be a ring, $X$ a scheme over $R$, $O_X$ the structure sheaf on $X$ and $i \in S$. We can construct $G_{m,n}$ and $H^n(X, G_{m,n})$ by $\text{NilR-alg}_R \to \text{sheaves of nil}-R$-algebra on $X$

$\text{sheaves of abelian groups on } X$

$H^n(X, G_{m,n})$ (commut.)

$H^s$

$\text{Nil}$-groups

where $O_X \otimes_R$ assigns to nil-$R$-algebra $A$ the sheaf $O_X \otimes_R A$ associated with the presheaf $U \mapsto \Gamma(U, O_X) \otimes_R A$; $G_{m,n}$ assigns to a sheaf $\mathcal{A}$ of nil-$R$-algebra the sheaves of abelian groups $G_{m,n}(\mathcal{A})$ defined by $\Gamma(U, G_{m,n}(\mathcal{A})) = \Gamma(U, G_{m,n}(\mathcal{A}))$ for every open $U \subset X$; $H^s$ is taking $s$-th cohomology.

The Honda's functor $H^s(X, G_{m,n})$ are called the Artin-Mazur functors.

**Formal Group Law for some A-M Functors** (Stienstra 1987)

Let $R$ be a noetherian ring, $F$ be a homogeneous polynomial in $R[T_0, \ldots, T_N]$ of degree $2d > 2N$ and let $X$ be the double covering of $\mathbb{P}^N$ defined by $W^2 = F(W)$ is a new variable of $\mathcal{W}$.

Then $H^n(X, G_{m,n})$ is a formal group over $R$ of dimension $n = \binom{d - 1}{N}$.

If $R$ is flat over $\mathbb{Z}$, put $J = \{ l = (l_0, \ldots, l_N) \in \mathbb{Z}^{N+1} | l_0, \ldots, l_N \geq 1, b_0 + \ldots + b_N = d \}$. Then there is a formal group law (the method III) for $H^n(X, G_{m,n})$ with logarithm $l(\tau) = (l_0, \ldots, l_N)$ given by

$l_i(\tau) = \sum_{m \geq 1} \sum_{j \geq 1} \tau_1^{m-1} \beta_{m,l_j} \tau_1^m$

where

$\beta_{m,l_j} = \begin{cases} 0 & \text{if } m \text{ is even; } \\ \left( \frac{\text{the coefficient of } T_0^{m-1} - T_0^{m-1} \cdot T_-^{m-1} \cdot \cdots \cdot T_0^{m-1} \cdot \cdots \cdot T_0^{m-1}}{\alpha_{m-1} \alpha_{m-1} \cdots \alpha_{m-1}} \right) & \text{if } m \text{ is odd.} \end{cases}$

*If $N = 1, d = 2$ and $F = T_0^2 + a_0 T_1 + b_0 T_1^2$, this is as same as the Weierstrass model of an elliptic curve $y^2 = x^3 + ax + b$. The logarithm $l(\tau)$ we got from this theorem is actually the logarithm of the standard group structure on the elliptic curve(!).