Homothetic Solution to Curve Shortening Flow

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Preliminaries

As a background of studying geometric flows, the simplest one is the curve shortening flow, given by one-parameter family of closed curves in the plane (\(t\) is time and \(s\) is curve parameter),

\[ \gamma(t) : [0, T] \times [0,1] \rightarrow \mathbb{R}^2, \text{ such that } \gamma(0) = \gamma(1) \]

Let \(N\) be unit normal vector, and \(\kappa\) is the curvature, then the evolution law \( \frac{d\kappa}{dt} = \kappa N \)

called the curve shortening flow (CSF).

Background of study

Following two results characterize the curve shortening flow (CSF) for embedded, closed initial curves:

- M. Gage and R. Hamilton [1]: Let \(\gamma(s)\) be a closed, convex curve, then the evolution law preserves convexity and shrinks to a point in a finite time.
- M. Grayson [2] showed that any closed, embedded curves become convex before it shrinks to a point

Aim

According to Abresch and Langer’s [3] study, we present curve shortening flow for plane curves which are not necessarily convex, and these curves evolve by homothety.

Modification of usual curve shortening flow

In [3], the curve shortening flow is modified by adding tangential field \(h T\) to \(k N\) in the evolution law, with this modification; flow is geometrically unchanged but this helps us to maintain the constant speed

Let \(M\) be 2-manifold and let \(\gamma(t) : [0, T] \times \mathbb{R}/\mathbb{Z} \rightarrow M\) be an evolving curve according to

\[ \frac{\partial\gamma}{\partial t} = h T + k N \]

where \(s \in \mathbb{R}/\mathbb{Z}\) is the curve parameter, and \(t \in [0, T]\) is the time parameter, \(N\) is the unit normal vector and \(k\) is the curvature.

Homothetic solution & main result

Contracting a homothetic solution of normalized CSF is a flow in which the shapes of the curves change homothetically and continuously to a point in finite time, which is called also contracting self-similar curve (the circle is a contracting self-similar curve). Moreover, other such curves must have self-intersections. In fact, all closed homothetically evolving curves classified by Abresch and Langer and it is represented in the following:

Proposition-2

A constant speed parametrized closed curve \(\gamma : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^2\) represents a homothetic solution of CSF if and only if its normalized curvature function \(\kappa\) (where \(\kappa(s, \tau) = \kappa(t, s)\))

\[ \frac{\partial\kappa}{\partial s} = k \kappa \quad \frac{\partial\kappa}{\partial \tau} = -k \kappa \quad \frac{\partial h}{\partial s} = \beta \]

where \(\beta(s, \tau)\) is an auxiliary function satisfies \(\beta(s, \tau) = \alpha(s) h(s, \tau),\) and \(0 < \lambda \in \mathbb{R}\).

Reformulation

Using (2) equations, we obtain \(\kappa = \lambda e^{-\frac{t^2}{2}}\)

rewriting \(\kappa\) as \(\kappa = \lambda e^{\frac{t^2}{2}}\) with \(\beta = -\frac{1}{2} e^t\)

substituting in (3), we obtain

\[ P'' + 2\lambda P = 0, \quad P = 2\log\left(\frac{\lambda}{\lambda}\right) \]

Theorem (Abresch - Langer)

Let \(\gamma : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^2\) be a unit speed closed curve representing a homothetic solution of the curve shortening flow, then \(\gamma\) is an \(m\)-covered circle or \(\gamma\) is a member of family of transcendental curves ([\(m, n]\)) having the following description: if \(m, n\) are positive integers satisfying \(1/2 < m/n < \sqrt{2}/2\), there is a unique unit speed curve \(\gamma_{m,n} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^2\) a solution to the equations

\[ P'' + 2\lambda P = 0, \quad P = 2\log\left(\frac{\lambda}{\lambda}\right) \]

for some constant \(\lambda\).