Asymptotic error distributions of the Crank-Nicholson scheme for SDEs driven by fractional Brownian motion

Nobuaki Naganuma
Mathematical Institute, Tohoku University

1 Introduction

We consider the following 1-dim SDE:

$$\begin{cases} dX_t = \sigma(X_t)\, d^\circ B_t, \quad t \in (0,1], \\ X_0 = x_0, \end{cases}$$

where B is a 1-dim. fBm on $(\Omega, \mathcal{F}, P)$ with the Hurst parameter $0 < H < 1$. The solution is expressed by $X_t = \phi(x_0, B_t)$.

Remark 1. FBm B is a conti. centered Gaussian proc. with

$$E[B_s B_t] = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H})$$

for some $0 < H < 1$. Note that

- B is a Bm if $H = 1/2$, else B is NOT a semimartingale.
- B is $(H - \epsilon)$-Hölder continuous.

The Crank-Nicholson scheme $\{\hat{X}^{(m)}\}_{m=1}^\infty$ is defined by the solution to the equation:

$$\begin{cases} \hat{X}_0^{(m)} = x_0 \\ \hat{X}_t^{(m)} = \hat{X}_{k2^{-m}}^{(m)} + \frac{1}{2}\left(\sigma\left(\hat{X}_t^{(m)}\right) + \sigma\left(\hat{X}_{k2^{-m}}^{(m)}\right)\right)(B_t - B_{k2^{-m}}), \\ \qquad t \in (k2^{-m}, (k+1)2^{-m}]. \end{cases}$$

2 Main theorem

Assumption 2. Assume $1/3 < H < 1/2$ and

$$\sigma \in C_{\mathrm{bdd}}^\infty(\mathbb{R};\mathbb{R}), \qquad \sup|\sigma'| > 0, \qquad \inf|\sigma| > 0.$$

Theorem 3 (N). *Under Assumption 2, we have*

$$\lim_{m\to\infty}\left(B, 2^{m(3H-1/2)}(\hat{X}^{(m)} - X)\right) = \left(B, \sigma(X_\cdot)\cdot c_{3,H}\int_0^\cdot f_3(X_s)\, dW_s\right)$$

weakly in $\mathcal{D}([0,1];\mathbb{R}^2)$, where $c_{3,H} > 0$, $f_3 = (\sigma^2)''/24$, and W is a standard Brownian motion independent of B.

3 Proof

We have 5 steps in order to prove the main theorem.

3.1 Analysis of the Hermite variations

Let $q \geq 2$ and $f \in C_{\mathrm{poly}}^{2q}(\mathbb{R};\mathbb{R})$. Put

$$G_q^{(m)}(t) = 2^{-m/2}\sum_{k=0}^{\lfloor 2^m t\rfloor - 1}\frac{f(B_{(k+1)2^{-m}}) + f(B_{k2^{-m}})}{2} \times H_q(2^{mH}\triangle B_{k2^{-m}}),$$

where H_q denotes the q-th Hermite polynomial.

Proposition 4 (N). *If $1/2q < H < 1 - 1/2q$, then we have*

$$\lim_{m\to\infty}(B, G_q^{(m)}) = \left(B, c_{q,H}\int_0^\cdot f(B_s)\, dW_s\right)$$

weakly in weakly in $\mathcal{D}([0,1];\mathbb{R}^2)$.

3.2 Expression of the Crank-Nicholson sheme

Proposition 5 ([1]). *Under Assumption 2, $\hat{X}^{(m)}$ satisfies*

$$\hat{X}_{k2^{-m}}^{(m)} = \phi\left(x_0, B_{k2^{-m}} + U_{k2^{-m}}^{(m)}\right) = X_{k2^{-m}} + \sigma(X_{k2^{-m}})U_{k2^{-m}}^{(m)} + O((U_{k2^{-m}}^{(m)})^2),$$

where $U^{(m)}$ is defined by $U_0^{(m)} = 0$ and

$$U_{(k+1)2^{-m}}^{(m)} = U_{k2^{-m}}^{(m)} + f_3\left(\hat{X}_{k2^{-m}}^{(m)}\right)(\triangle B_{k2^{-m}})^3 + f_4\left(\hat{X}_{k2^{-m}}^{(m)}\right)(\triangle B_{k2^{-m}})^4 + O\left((\triangle B_{k2^{-m}})^5\right)$$

where $f_3 = (\sigma^2)''/24$ and $f_4 = \sigma(\sigma^2)'''/48$.

3.3 Decomposition into the main term and the remainders

Proposition 6 (N). *Under Assumption 2, we have the expansion, for every $\alpha \geq 1$,*

$$U^{(m)} = \sum_{\beta=1}^{\alpha}\Phi^{(m,\beta)} + O(2^{m(\alpha+1)}(\triangle B)^{3(\alpha+1)}),$$

where $\Phi^{(m,1)}$ is definde by $\Phi_0^{(m,1)} = 0$ and

$$\Phi_{(k+1)2^{-m}}^{(m,1)} = \Phi_{k2^{-m}}^{(m,1)} + f_3(X_{k2^{-m}})(\triangle B_{k2^{-m}})^3 + f_4(X_{k2^{-m}})(\triangle B_{k2^{-m}})^4,$$

and, for $\beta \geq 2$, $\Phi^{(m,\beta)}$ is also defined explicitly.

3.4 Convergence of the main term

Proposition 7 (N). *Under Assumption 2, we have*

$$\lim_{m\to\infty}(B, 2^{m(3H-1/2)}\Phi^{(m,1)}) = \left(B, c_{3,H}\int_0^\cdot h(B_s)\, dW_s\right)$$

weakly in $\mathcal{D}([0,1];\mathbb{R}^2)$.

Proof. Put $h(\eta) = f_3(\phi(x_0,\eta))$. Then we have

$$2^{m(3H-1/2)}\Phi^{(m,1)} = 2^{-m/2}\sum_{k=0}^{\lfloor 2^m\cdot\rfloor - 1}\left(h(B_{k2^{-m}}) + \frac{1}{2}h'(B_{k2^{-m}})\right)(2^{mH}\triangle B_{k2^{-m}})^3$$

Using the Taylor formula, $\xi^3 = H_3(\xi) + 3\xi$ and Proposition 4, we have the assertion. □

3.5 Convergence of the remainders

By long calculation, we have $\Phi^{(m,\beta)} \to 0$ for $\beta \geq 2$.

References

[1] I. Nourdin. A simple theory for the study of SDEs driven by a fractional Brownian motion, in dimension one. In *Séminaire de probabilités XLI*, volume 1934 of *Lecture Notes in Math.*, pages 181–197. Springer, Berlin, 2008.