Ultracontractivity for Markov semigroups and quasi-stationary distributions

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Notation
- $E$: locally compact separable metric space
- $X_t$: $m$-symmetric Hunt process
- $L^p(m) := \{ f : \int_E |f|^p \, dm < \infty \}$
- $\{T_t\}$: the transition semigroup of $X_t$

A probability measure $\mu$ is a quasi-stationary distribution (QSD) of $X_t$
$$\def\P{\mathbb{P}} \P(\mu(X_t \in B \mid t < \zeta) = \mu(B), \ \forall t > 0$$

Assumption
- The transition density $p_t(\cdot, \cdot)$ exists, i.e., $T_t f(x) = \int_E f(y) p_t(x, y) m(dy)$
- $\exists \lambda > 0, \exists \phi \in L^2(m)$: ground state
  i.e., $T_t \phi(x) = e^{-\lambda t} \phi(x)$

$X_t$: intrinsically ultracontractive (IU)
$$\def\P{\mathbb{P}} \P(\exists t > 0, \exists \alpha_t, \beta_t > 0 \text{ s.t.,} \quad \alpha_t \leq \frac{p_t(x, y)}{\phi(x) \phi(y)} \leq \beta_t, \ x, y \in E$$

Remark
$$\{T_t\} : 1U \Rightarrow \phi \in L^1(m)$$

Key lemma
Assume $m(E) < \infty$ and $\{T_t\}$: conservative.
Then for $\forall f \in L^1(E; m)$,
$$\lim_{t \to \infty} T_t f(x) - (m(E))^{-1} \int_E f \, dm, \ m\text{-a.e.}$$

Theorem
$$T_t: 1U \Rightarrow \mu(B) := \frac{\int_E \phi \, dm}{\int_E \phi \, dm} \text{ is a QSD of } X_t$$
(p: $T_t^\phi f(x) := \frac{e^{\lambda t}}{\phi(x)} T_t(\phi f)(x)$
$$\Rightarrow \{T_t^\phi\}$ semigroup on $L^2(\phi^2 m)$ satisfying the assumption in Key lemma
$$\lim_{t \to \infty} \frac{P_x(X_t \in B; t < \zeta)}{P_x(t < \zeta)} = \lim_{t \to \infty} \frac{T_t^\phi(1/\phi)(x)}{T_t^\phi(1/\phi)(x)} \uparrow \frac{\int_B \phi \, dm}{\int_E \phi \, dm} = \mu(B)$$
Key lemma
$$\Rightarrow \mu \text{ is a QSD.}$$

Example
Logistic model: $dZ_t = \sqrt{Z_t} \, dB_t \mid r(Z_t - K Z_t^2) \, dt, \ Z_0 = z > 0$

Set $X_t := 2 \sqrt{Z_t}$, Itô’s formula
$$dX_t = dB_t - q(X_t) \, dt \left( q(u) = \frac{1}{2u} - \frac{ru}{2} + \frac{r^2 K u^3}{8} \right)$$
$$\sim X_t: \text{diffusion on } (0, \infty) \text{ with the speed measure } m \text{ and the scale function } s$$
where $dm(x) = e^{-Q(x)} \, dx$, $s(x) = \int_x^\infty e^{Q(y)} \, du$ \quad $Q(x) = 2 \int_1^x q(u) \, du$\quad classify boundary points

0: exit boundary, \infty: entrance boundary \quad (\Rightarrow X_t \text{ satisfies Assumption (a), (b)})

\downarrow \quad \text{check Tomisaki's condition} \quad \text{\ldots} \quad \text{[a sufficient condition for IU in terms of the speed measure and the scale function]}

$X_t$: IU $\Rightarrow$ A QSD of $Z_t$ exists