On vorticity formulation for viscous incompressible flows in the half plane and its application to the inviscid limit problem

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1. Introduction. I

To study the flows of the high Reynolds number is an important theme in the fluid mechanics. Indeed, for many typical fluids such as water and air, the kinematic viscosity is very small (water: $1.0 \times 10^{-6} m^2/s$ at $20^\circ C$).

Such flows may be considered as the ideal flows, whose motion is described by the Euler equations.

However, in the presence of the boundary, it is known that the Euler equations for the incompressible flows do not precisely predict the behavior of the real flows, especially near the boundary.

This is due to neglecting the viscosity effects. The ideal fluid slips along the wall, while a real fluid adheres to a solid wall and tangential forces do act.
1. Introduction. II

So the theory of the Navier-Stokes equations or the boundary layer theory, which takes the viscosity effects into account, has a great importance in the presence of the boundary.

Nevertheless, the mathematically rigorous description of the fluid motions at the high Reynolds number limit is still widely open even for two-dimensional flows if the boundary is present, in spite of many important contributions by numerical researches or experiments.

In this talk we are interested in the vorticity of the flows, which is defined as the curl of the velocity and plays important roles in local structures of the flows.

Compared with NS, the mathematical study of the vorticity equations is still undeveloped if nontrivial boundary exists, because of the complicated boundary condition which the vorticity should satisfies.

Thus we focus on the simplest case here, that is, the case of the half plane.
1-1. Two-dim. incompressible Navier-Stokes equations

\[
\begin{align*}
\partial_t u_i + u \cdot \nabla u_i - \nu \Delta u_i + \partial_{x_i} p &= 0 \quad t > 0, \quad x \in \Omega, \\
\partial_{x_1} u_1 + \partial_{x_2} u_2 &= 0 \quad t \geq 0, \quad x \in \Omega, \\
u > 0; \text{kinematic viscosity coefficient} \\
\Omega; \text{a domain in } \mathbb{R}^2 \text{ with smooth boundary } \partial \Omega
\end{align*}
\]

\[u = u(t, x) = (u_1(t, x), u_2(t, x)); \text{velocity field} \]
\[p = p(t, x); \text{pressure field} \]
\[u \cdot \nabla f = u_1 \partial_{x_1} f + u_2 \partial_{x_2} f, \quad \Delta f = \partial_{x_1}^2 f + \partial_{x_2}^2 f\]
1-2. Energy relation

By performing the integration by parts based on the no-slip boundary condition (i.e. \( u = 0 \) on \( \partial \Omega \)) we formally obtain the key equality for the kinetic energy \( \| u(t) \|_{L^2}^2 = \int_{\Omega} |u(t, x)|^2 \, dx \):

\[
\frac{1}{2} \frac{d}{dt} \| u(t) \|_{L^2}^2 = - \nu \| \nabla u(t) \|_{L^2}^2 \quad t > 0,
\]

that is,

\[
\| u(t) \|_{L^2}^2 + 2 \nu \int_0^t \| \nabla u(s) \|_{L^2}^2 \, ds = \| a \|_{L^2}^2 \quad t \geq 0. \quad (1)
\]
1-3. Solvability of 2D Navier-Stokes equations. I

The energy equality (1) gives the a priori bound of

\[ \sup_{t>0} \| u(t) \|_{L^2}^2, \int_0^\infty \| \nabla u(t) \|_{L^2}^2 \, dt, \]

which is essential to solve (NS) in the energy space.
1-3. Solvability of 2D Navier-Stokes equations. II

It is known that, for any initial velocity $a$ with finite energy, there is a unique and smooth solution $(u, p)$ to (NS) in the energy space $L^\infty(0, \infty; L^2_\sigma(\Omega)) \cap L^2(0, \infty; \dot{H}^1_0(\Omega)^2)$;

Leray (’33,’34), Hopf (’50), Ladyzhenskaya (’59), Masuda (’84), Borchers-Miyakawa (’92), Kozono-Ogawa (’93), · · ·.

There is another approach to solve (NS), which uses the abstract operator theory;

Kato-Fujita (’62), Kato (’84), Giga-Miyakawa (’85), · · ·.
1-4. Vorticity transport equations

The vorticity is defined by $\omega = \text{Rot} u := \partial_{x_1} u_2 - \partial_{x_2} u_1$.

Recalling the relations $\text{Rot} (u \cdot \nabla u) = u \cdot \nabla \omega$, $\text{Rot} (\nabla p) = 0$, we get

$$\frac{\partial}{\partial t} \omega - \nu \Delta \omega + u \cdot \nabla \omega = 0 \quad t > 0, \quad x \in \Omega. \tag{2}$$

When $\Omega = \mathbb{R}^2$ we have the maximum principle

$$\max_{x \in \mathbb{R}^2} |\omega(t, x)| \leq \max_{x \in \mathbb{R}^2} |b(x)| \quad b = \text{Rot} a,$$

or more generally, thanks to $\text{div} u = 0$, we have

$$\|\omega(t)\|_{L^p(\mathbb{R}^2)} \leq \|b\|_{L^p(\mathbb{R}^2)} \quad 1 \leq p \leq \infty. \tag{3}$$
1-5. Solvability of vorticity transport equations in $\mathbb{R}^2$

In $\mathbb{R}^2$ the velocity $u$ is recovered from the Biot-Savart law:

$$u(t, x) = \nabla^\perp (-\Delta_{\mathbb{R}^2})^{-1} \omega(t)(x), \quad \nabla^\perp = (\partial_{x_2}, -\partial_{x_1}),$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(t, y) \, dy, \quad x^\perp = (-x_2, x_1).$$

The estimate (3) gives the a priori estimate of vorticity fields. In particular, the solvability of (2) and the property of solutions are well known when $\Omega = \mathbb{R}^2$;

Giga-Miyakwa-Osada ('88), Ben-Artzi ('94), Kato ('94), Gallay-Wayne ('05), · · ·.
1-6. Theme of talk: Vorticity equations in the half plane

In the presence of boundary when the no-slip boundary condition is imposed on the velocity fields, there are still only a few mathematical results on the vorticity equations, mainly because of the boundary conditions on the vorticity fields.

However, it is numerically or experimentally observed that the vorticity fields play important roles in the fluid dynamics even near the boundary.

Indeed, the vorticity is supposed to be created on the boundary, and it becomes more vigorous as the Reynolds number is increasing.

The interaction between the vorticity structures inside the domain and the viscous boundary layer at the no-slip boundary has been one of the most important subjects in the fluid mechanics.

In this talk we consider the simplest case, i.e., the case of the flat boundary, and present some mathematical results.
2-1. Vorticity boundary conditions

Although only a little attention has been paid so far, it is known that vorticity fields do satisfy certain boundary conditions (Anderson ’89). Indeed, when $\Omega = \mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 > 0\}$ the vorticity $\omega$ must satisfy the following boundary condition:

$$v(\partial_{x_2} + (-\partial^2_{x_1})^{1/2})\omega = -\partial_{x_2}(-\Delta_D)^{-1}(u \cdot \nabla \omega) \quad t > 0, \quad x \in \partial \mathbb{R}^2_+.$$  \hspace{1cm} (4)

Here $f = (-\Delta_D)^{-1}h$ denote the solution to the Poisson equations:

$-\Delta f = h$ in $\mathbb{R}^2_+$, $f = 0$ on $\partial \mathbb{R}^2_+$.

The operator $(-\partial^2_{x_1})^{1/2}$ is defined in terms of the Fourier transform $\mathcal{F}$ in the tangential $(x_1)$ direction:

$$(-\partial^2_{x_1})^{1/2}f(x_1) = \mathcal{F}^{-1}[[|\xi_1|\mathcal{F}[f](\xi_1)](x_1). \quad (5)$$
2-2. Vorticity boundary condition: derivation

The condition (4) is derived from a simple mathematical consideration using the Biot-Savart law in $\mathbb{R}^2_+$:

\[ u(t) = \nabla^\perp (\Delta_D)^{-1} \omega(t) \quad \nabla^\perp = (\partial_{x_2}, -\partial_{x_1}). \quad (6) \]

By (6) the condition $u_2(t) = 0$ on $\partial \mathbb{R}^2_+$ is automatically satisfied. We need $u_1(t) = 0$ on $\partial \mathbb{R}^2_+$, that is,

\[ u_1(t, x_1, 0) = \frac{1}{2\pi} \int_{\mathbb{R}^2_+} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega(t, y) \, dy = 0. \quad (7) \]

However, (7) is highly non-local and difficult to adopt as a boundary condition on the vorticity. Thus we impose the condition so that (7) is preserved under the evolution of the vorticity in $\mathbb{R}^2_+$, i.e., the vorticity equations (2).
2-3. Vorticity boundary conditions: enstrophy evolution

From energetic point of view, the term $\nu (-\partial_{x_1}^2)^{1/2} \omega$ indicate a linear creation of vorticity on the boundary:

\[
\frac{1}{2} \frac{d}{dt} \|\omega(t)\|_{L^2(\mathbb{R}_+^2)}^2 = -\nu \|\nabla \omega(t)\|_{L^2(\mathbb{R}_+^2)}^2 + \nu \|(-\partial_{x_1}^2)^{1/4} \omega(t)\|_{L^2(\partial \mathbb{R}_+^2)}^2 \\
+ \langle \omega, \partial_{x_2} (-\Delta_D)^{-1} (u \cdot \nabla \omega) \rangle_{L^2(\partial \mathbb{R}_+^2)}.
\]

Here

\[
\|f\|_{L^2(\partial \mathbb{R}_+^2)}^2 = \langle f, f \rangle_{L^2(\partial \mathbb{R}_+^2)} = \int_{\mathbb{R}} |f(x_1, 0)|^2 \, dx_1,
\]

and we have used the relation

\[
\langle \omega, (-\partial_{x_1}^2)^{1/2} \omega \rangle_{L^2(\partial \mathbb{R}_+^2)} = \|(-\partial_{x_1}^2)^{1/4} \omega(t)\|_{L^2(\partial \mathbb{R}_+^2)}^2.
\]
2-4. Remark on enstrophy evolution

Since the inequality \( \|(-\partial_{x_1}^2)^{1/4} f\|_{L^2(\partial \mathbb{R}^2_+)} \leq \|\nabla f\|_{L^2(\mathbb{R}^2_+)} \) holds in general, we observe from (8) that

\[
\frac{1}{2} \frac{d}{dt} \|\omega(t)\|_{L^2(\mathbb{R}^2_+)}^2 \leq \langle \omega(t), \partial x_2 (-\Delta_D)^{-1}(u \cdot \nabla \omega(t)) \rangle_{L^2(\partial \mathbb{R}^2_+)}.
\]

Hence the amplification of the enstrophy is essentially governed by the non-local and nonlinear fluctuation on the boundary given by the right-hand side of the above inequality.
3. Aim of this talk
In this talk we consider the IBP for the vorticity equations:

\[
\begin{align*}
\frac{\partial}{\partial t} \omega - \nu \Delta \omega + u \cdot \nabla \omega &= 0 \\
\omega|_{t=0} &= b := \text{Rot } a \\
\end{align*}
\]

with the boundary conditions

\[
\nu \left( \frac{\partial}{\partial t} + \left( -\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} \right) \omega = -\frac{\partial}{\partial x} \left( -\Delta_D \right)^{-1} (u \cdot \nabla \omega) \\
\nu \left( \frac{\partial}{\partial t} + \left( -\frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} \right) \omega = -\frac{\partial}{\partial x} \left( -\Delta_D \right)^{-1} (u \cdot \nabla \omega) \\
\]

(i) Solution formula and local-in-time solvability
(ii) Application to the zero viscosity limit
3-1. Solution formula for linearized problem

Let us consider the linear problem

\[
\begin{aligned}
\frac{\partial}{\partial t} \omega - \nu \Delta \omega &= f \\
\omega|_{t=0} &= b
\end{aligned}
\quad t > 0, \quad x \in \mathbb{R}_+^2, \quad (LV)
\]

subject to the boundary conditions

\[
\nu (\partial_2 + (-\partial_1^2)^{\frac{1}{2}}) \omega = g \\
\quad t > 0, \quad x \in \partial \mathbb{R}_+^2. \quad (LBC)
\]
3-1. Solution formula for linearized problem

Let $G(t, x)$ be the Gauss kernel in $\mathbb{R}^2$, and $E(x)$ be the Newton potential in $\mathbb{R}^2$, i.e.,

$$G(t, x) = \frac{1}{4\pi t} \exp\left(-\frac{|x|^2}{4t}\right), \quad E(x) = -\frac{1}{2\pi} \log |x|.$$ 

The following notations will be used:

$$(h_1 * h_2)(x) = \int_{\mathbb{R}^2_+} h_1(x - y)h_2(y) \, dy,$$

$$(h_1 \star h_2)(x) = \int_{\mathbb{R}^2_+} h_1(x - y^*)h_2(y) \, dy \quad y^* = (y_1, -y_2),$$

$$h * (g\mathcal{H}^1_{\{\partial \mathbb{R}^2_+\}})(x) = h \star (g\mathcal{H}^1_{\{\partial \mathbb{R}^2_+\}})(x) = \int_{\mathbb{R}} h(x_1 - y_1, x_2)g(y_1) \, dy_1.$$
3-1. Solution formula for linearized problem

Then we set

\[
\Gamma(t, x) = 2 \left( \partial_{x_1}^2 + \left(-\partial_{x_1}^2\right)^{\frac{1}{2}} \partial_{x_2} \right) (E \ast G(t))(x),
\]
\[
e^{t\Delta_N} f = G(t) \ast f + G(t) \star f.
\]

**Remark.** (i) \( \theta(t) = e^{vt\Delta_N} f \) defines the solution to the heat equations in \( \mathbb{R}^2_+ \) subject to the homogeneous Neumann boundary condition with the initial data \( f \):

\[
\partial_t \theta - \nu \Delta \theta = 0 \quad \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2_+,
\]
\[
\partial_{x_2} \theta = 0 \quad \text{for } (t, x) \in \mathbb{R}_+ \times \partial \mathbb{R}^2_+.
\]

(ii) \( \Gamma(0) \star f := \lim_{\epsilon \downarrow 0} \Gamma(\epsilon) \star f = 2 \left( \partial_{x_1}^2 + \left(-\partial_{x_1}^2\right)^{\frac{1}{2}} \partial_{x_2} \right) E \star f \) satisfies

\[
(\partial_{x_2} + \left(-\partial_{x_1}^2\right)^{\frac{1}{2}}) \Gamma(0) \star f = 0 \quad \text{on } \overline{\mathbb{R}^2_+}.
\]
Theorem 1. The integral representation for solutions to (LV)-(LBC) is given by

\[ \omega(t) = e^{\nu t B} b - \Gamma(0) \star b + \int_0^t e^{\nu(t-s)B}(f(s) - g(s)H^1_{\partial \mathbb{R}^2_+}) \, ds \]

\[ - \int_0^t \Gamma(0) \star (f(s) - g(s)H^1_{\partial \mathbb{R}^2_+}) \, ds. \]

Here \( e^{\nu t B} \) is defined by

\[ e^{\nu t B} f = e^{\nu t \Delta_N} f + \Gamma(\nu t) \star f. \]

Remark. For (NS) the solution formula is obtained by Solonnikov (’68) and Ukai (’87).
Since

\[ \Gamma(0) \star f = 2(\partial_1^2 + (-\partial_1^2)^{\frac{1}{2}} \partial_2)E \star f \quad E(x) = -\frac{1}{2\pi} \log |x|, \]

the operator \( \Gamma(0) \star \) does not have a smoothing effect near the boundary. In fact, this term does not appear in the vorticity equations, due to the following cancellation property.

**Proposition 1.** If \( g = \partial_2(-\Delta_D)^{-1}f \mid_{x_2=0} \) then

\[ \Gamma(0) \star (f - g\mathcal{H}_{\{x_2=0\}}) = 0 \text{ in } \mathbb{R}_+^2. \]

In particular, we have \( \Gamma(0) \star b = 0 \text{ in } \mathbb{R}_+^2 \) if \( \partial_2(-\Delta_D)^{-1}b = 0 \text{ on } \partial \mathbb{R}_+^2 \).

Note that the condition \( \partial_2(-\Delta_D)^{-1}b = 0 \text{ on } \partial \mathbb{R}_+^2 \) is nothing but the compatibility condition: \( a_1 = 0 \text{ on } \partial \mathbb{R}_+^2 \).
3-2. $L^p - L^q$ estimates for $e^{tB}$

Lemma 1. (i) Let $1 \leq q < p \leq \infty$ or $1 < q \leq p < \infty$. Then

$$\|e^{tB} f\|_{L^p} \leq Ct^{-\frac{1}{q} + \frac{1}{p}} \|f\|_{L^q} \quad t > 0.$$ 

(ii) Let $1 \leq q \leq p \leq \infty$ and $p > 1$. Then

$$\|e^{tB}(gH_{\{x_2=0\}}^1)\|_{L^p} \leq Ct^{-\frac{1}{2}(1 + \frac{1}{p} - \frac{2}{q})} \|g\|_{L^q_{x_1}} \quad t > 0.$$ 

(iii) Let $1 \leq q \leq p \leq \infty$ and $k \in \mathbb{N}$. Then

$$\|\nabla^k e^{tB} f\|_{L^p} \leq Ct^{-\frac{1}{q} + \frac{1}{p} - \frac{k}{2}} \|f\|_{L^q} \quad t > 0.$$ 

(iv) Let $1 \leq q \leq p \leq \infty$ and $g = \partial_2 (\Delta)^{-1} f \mid_{x_2=0}$. Then

$$\|e^{tB}(f - gH_{\{x_2=0\}}^1)\|_{L^p} \leq Ct^{-\frac{1}{q} + \frac{1}{p} - \frac{1}{2}} \|\nabla^\perp (\Delta_D)^{-1} f\|_{L^q} \quad t > 0.$$
3-3. Solvability of IBP for the vorticity equations in $\mathbb{R}^2_+$

**Theorem 2.** Assume that $b \in L^p(\mathbb{R}^2_+)$, $\exists p \in (1, 2)$, and that $b$ satisfies $\partial_2(-\Delta_D)^{-1}b = 0$ on $\partial\mathbb{R}^2_+$. Then there is $T > 0$ such that (V)-(BC) has a unique mild solution $\omega \in C([0, T); L^p)$ satisfying
\[
\sup_{0 < t < T} t^{1/p-1/4} \|\omega(t)\|_{L^4} < \infty.
\]
Furthermore, the solution is smooth in positive time.

**Remark.** (i) When $\Omega = \mathbb{R}^2$ the solvability of the vorticity equations is classical; Giga-Miyakawa-Osada ('88), Ben-Artzi ('94), Kato ('94).

(ii) In view of the solvability of the Navier-Stokes equations, Theorem 2 does not give a new result. For (NS) the $L^p$ theory is already well developed; Solonnikov ('77), Weissler ('80).
4. Analysis of vorticity at inviscid limit

The inviscid limit behavior of solutions to (NS) is a classical theme in fluid mechanics.

However, if the no-slip boundary conditions are imposed on velocity fields, only partial results are known even in the two-dimensional case, because of the appearance of the boundary layer.
4-1. Recall: Formal asymptotics - the Euler equations

Formally, by tending $\nu \to 0$ in (NS) we get the Euler equations for the ideal incompressible flows

$$\begin{cases}
\partial_t u_E + u_E \cdot \nabla u_E + \nabla p_E = 0 & t > 0, \quad x \in \mathbb{R}^2_+,
\text{div} \ u_E = 0 & t \geq 0, \quad x \in \mathbb{R}^2_+,
ue_{1,2} = 0 & t \geq 0, \quad x \in \partial \mathbb{R}^2_+,
u_E|_{t=0} = a & x \in \mathbb{R}^2_+.
\end{cases}$$

We recall that $u_{NS} = 0$ on $\partial \mathbb{R}^2_+$, while $u_{E,1} \neq 0$ on $\partial \mathbb{R}^2_+$ in general. $\Rightarrow$ The boundary layer arises in the inviscid limit.
4-2. Recall: Formal asymptotics - Prandtl’s argument

· Prandtl’s idea (1904)

For the high Reynolds number flow ($0 < \nu \ll 1$) the domain is decomposed into the following two regions.

(i) Outer region (the region away from the boundary):
The fluid motion will be described by the Euler equations.

(ii) Boundary layer (the region where the viscosity effect essentially exists):
The boundary layer thickness is formally estimated as $O(\nu^{1/2})$. 
4-3. Recall: Formal asymptotics - the Prandtl equations
By assuming the expansion at $\nu \to 0$ as

$$u_{NS}(t, x) = u_E(t, x) + u_P^{(\nu)}(t, x) + \text{remainder},$$

$$u_P^{(\nu)}(t, x) = (v_{P,1}(t, x_1, x_2/\nu^2), \nu^2 v_{P,2}(t, x_1, x_2/\nu^2)),$$

we get the Prandtl equations for $v_P(t, x_1, X_2)$:

$$\begin{aligned}
&\left(\partial_t - \partial_{X_2}^2\right) v_{P,1} + v_{P,1} \partial_{x_1} v_{P,1} + v_{P,2} \partial_{X_2} u_{P,1} + \partial_{x_1} \pi_P = 0, \\
&\partial_{x_1} v_{P,1} + \partial_{X_2} v_{P,2} = 0, \quad \partial_{X_2} \pi_P = 0, \\
&v_P \mid_{X_2=0} = 0, \\
&\lim_{X_2 \to \infty} v_{P,1}(t, x_1, X_2) = u_{E,1}(t, x_1, 0), \\
&\lim_{X_2 \to \infty} \pi_P(t, x_1, X_2) = p_E(t, x_1, 0).
\end{aligned}$$
4-4. Known mathematical results

So far the solvability of the Prandtl equations and the verification of the expansion (9) are established only under restrictive situations.

(1) Mathematical analysis of the Prandtl equations
Oleinik ('66), Matsui-Shirota ('84); Monotonic data
Sammartino-Caflisch ('98); Analytic data
Lombardo et al. ('03); Analytic (in $x_1$ direction) data

(2) Verification of the asymptotic expansion (9)
Asano ('88), Sammartino-Caflisch ('98); Analytic initial data

(3) Counter example for (9)
Grenier ('00); (9) does not hold at least around the linearly unstable shear layer profile for the stationary Euler equations
4-5. Known results for $L^2$ convergence

$$\lim_{\nu \to 0} \left\| u^{(\nu)}_{NS}(t) - u_E(t) \right\|_{L^2(\Omega)} = 0. \quad (10)$$

(1) Under the radial symmetry of the domain and the solution the convergence (10) is verified.

Matsui (’94), LopesFilho et al (’08), Kelliher (’08), · · ·

(2) Criterion on the convergence (10) is known.

Kato (’84), Temam-Wang (’97), Kelliher (’08,’09), · · ·

$$\lim_{\nu \to 0} \int_0^T \nu \left\| \nabla u(t) \right\|_{L^2(\Omega_{\nu})}^2 \, dt = 0, \quad \Omega_{\nu} = \{ x \in \Omega \mid \text{dist} (x, \partial \Omega) \leq \nu \}.$$
4-6. Inviscid limit for initial vorticity with compact support

**Goal:** Establish the asymptotic expansion (for a short time) when the support of the initial vorticity is away from the boundary.

This class of initial data includes a dipole vortex as a typical example, which is used as a benchmark in the study (numerical or experimental one) of the interaction between the vorticity originated inside the domain and the vorticity created on the boundary; cf. Orlandi (’90).
FIG. 2. Sequence of vorticity contour plots showing the flow evolution of a dipole colliding with a no-slip wall for integral-scale Reynolds number Re = 2500. The contour levels are drawn for ..., -100, -60, -20, 20, 60, 100, ...
Extract from Nguyen-Farge-Schneider (Phys.Rev.Lett. 2011): (the horizontal direction is $\rightarrow$)

FIG. 1 (color online). Vorticity in the subdomain $[0.708, 0.962] \times [0.5, 0.754]$ at $t = 0.36$, 0.4, 0.45, and 0.495 (left to right) for $Re = 7880$. The white dotted box at $t = 0.495$ frames region $B$ (see text). Black pixels correspond to $\omega = \pm 300$ in all pictures.
Rough description of the behavior of vorticity with a dipole initial vortex

1. The dipole vortex approaches to the boundary, while the high vorticity is immediately created along the boundary (vortex line).

2. The dipole vortex collides into the boundary and the produced vortex line starts to roll up.

3. The produced vortex pairs with the original one and forms a secondary dipole.

4. The secondary dipole bounces back to the boundary.

Note. For the Euler flows the vorticity is independent of time along the trajectory flows (the Lagrange theorem) and the rebound of the secondary dipole vortex is not observed; Saffman (’79).

Prandtl’s theory will be applied only for Step 1.
4-7. Formal asymptotics - vorticity for the Euler flows

By taking the formal limit $\nu \to 0$ in (V) we get

$$\partial_t \omega_E + u_E \cdot \nabla \omega_E = 0, \quad u_E = \nabla^\perp (-\Delta_D)^{-1} \omega_E, \quad \omega_E|_{t=0} = b,$$

(V_E)

cf.) The solvability of the two-dimensional Euler equations for the incompressible flows is well known,

Wolibner ('33), Yudovich ('63), Kato ('67), Bardos ('72), ...
4-8. Formal asymptotics - vorticity for boundary layer

Assume: \( \omega^{(v)}(t, x) = \omega_E(t, x) + v^{-\frac{1}{2}}w_P(t, x_1, x_2/v^2) + \text{remainder.} \)

\[
\begin{align*}
\partial_tw_P - \partial_{x_2}^2w_P &= -(\nu_{E,1} + \nu_{P,1})\partial_1w_P - (\nu_{E,2} + \nu_{P,2})\partial_2w_P, \\
\nu_{E,1}(t, x_1, x_2) &= u_{E,1}(t, x_1, 0), \\
\nu_{E,2}(t, x_1, x_2) &= X_2\partial_2u_{E,2}(t, x_1, 0), \\
\nu_{P,1}(t, x_1, x_2) &= \int_{X_2}^\infty w_P(t, x_1, Y_2) \, dY_2, \\
\nu_{P,2}(t, x_1, x_2) &= -\partial_1\bigg(\int_0^{X_2} Y_2w_P(t, x_1, Y_2) \, dY_2 \\
&\quad + X_2\int_{X_2}^\infty w_P(t, x_1, Y_2) \, dY_2\bigg), \\
\big|w_P\big|_{t=0} &= 0,
\end{align*}
\]

subject to the boundary condition

\[
\partial_2w_P = -\int_0^\infty (\nu_E + \nu_P) \cdot \nabla w_P \, dY_2 - \partial_2(-\Delta_D)^{-1}(u_E \cdot \nabla \omega_E).
\]
4-9. Main result

**Theorem 3.** Let \( b = \text{Rot} \ a \in W^{4,1}(\mathbb{R}^2_+) \cap W^{4,2}(\mathbb{R}^2_+) \) satisfy the compatibility condition \( a = 0 \) on \( \partial \mathbb{R}^2_+ \). Assume that

\[
d_0 = \text{dist} (\partial \mathbb{R}^2_+, \text{supp} \ b) > 0.
\]

Then there is \( T > 0 \) such that the solution \( \omega^{(v)} \) to (V)-(BC) is decomposed as

\[
\omega^{(v)}(t, x) = \omega_E(t, x) + \frac{1}{v^2} w_P(t, x_1, \frac{x_2}{v^2}) + O(v^{\frac{1}{2}}) \quad 0 < t \leq T
\]

in a suitable function space. The time \( T \) is estimated from below as

\[
T \geq c \min\{d_0, 1\}, \quad \text{where } c > 0 \text{ depends only on } \|b\|_{W^{4,1} \cap W^{4,2}}.
\]
4-10. Remark on Theorem 3

(1) In the proof of Theorem 3 it is also shown that

\[ \| u^{(y)}(t) - u_E(t) - u_P^{(y)}(t) \|_{L^\infty} \leq C\nu^{\frac{1}{2}} \quad 0 < t \leq T. \]  \hspace{1cm} (13)

Sammartino-Caflisch ('98) proved (13) for the analytic initial data from the Navier-Stokes equations.

(2) Our solution is analytic only near the boundary. Hence, at least the boundary layer equation (in the vorticity form) is expected to be solved for a short time by known arguments.

However, even if the boundary layer equation is solved, the expansion (13) is not trivial; cf. Grenier’s counter example ('00).

(3) The lower bound of \( T \) gives an information of time such that the vortex line keep stable and does not separate beyond the classical boundary layer thickness.
4-11. Key observation for the proof of Theorem 3

(1) Since the vorticity field $\omega_E$ of the Euler equations solves the transport equation we have

$$\bigcup_{0<t<T'} \operatorname{supp} \omega_E(t) \subset \{x \in \mathbb{R}^2_+ \mid x_2 \geq 32d_E > 0\}, \quad (14)$$

for some $T' > 0$ and $0 < d_E < \min\{d_0, 1\}$. In particular, all data in the vorticity (NS) equations are analytic near the boundary layer region.

$\implies$ The Prandtl type equations should be solved (construction of the boundary layer $\omega_{B_{\nu}}$).
(2) Also for the remainder part $\omega_{I,v} = \omega - \omega_E - \omega_{B,v}$, we can use the analyticity near the boundary, while we have to work in the Sobolev class away from the boundary.

But in the region away from the boundary, we expect that the arguments for the vorticity equations in $\mathbb{R}^2$ can be applied to some extent.

(3) Due to the support property of $\omega_E$, there should be the region

$$D_{\text{small}} = \{ x \in \mathbb{R}^2_+ \mid c_1 d_E \leq x_2 \leq c_2 d_E \},$$

where $\omega_{I,v}$ is exponentially small, say,

$$|\omega_{I,v}(t, x)| \leq C \exp \left( -\frac{c}{\nu} \right) \quad \text{in} \quad D_{\text{small}}.$$
4-12. Difficulty and Key idea

**Difficulty**: How to capture the properties (1) - (3) rigorously by taking into account the interaction between the vorticity inside the domain and the boundary layer.

**Key idea**

(1) Introduce suitable weighted spaces

(2) Introduce a suitable decomposition for the remainder part \( \omega_{I_v} = \omega - \omega_E - \omega_{B_v} \)

- The boundary part \( \omega_{IB_v} = R_v w_{IB_v} \)
- The outer part \( \omega_{II_v} = w_{II_v} \)

Here \( R_v \) is the scaling operator defined by

\[
(R_v f)(x) = v^2 f(x_1, v^2 x_2).
\]
(3) Use optimal pointwise estimates by Carlen-Loss (’95) for fundamental solutions to

\[ \partial_t \theta - \nu \Delta \theta + u \cdot \nabla \theta = 0 \quad \nabla \cdot u = 0, \quad t > 0, \quad x \in \mathbb{R}^2. \]

\[ P_u^{(\nu)}(t, x; s, y) \leq \frac{1}{4\pi \nu (t - s)} \exp \left( - \frac{(|x - y| - \int_s^t \|u(\tau)\|_{L^\infty} \, d\tau)_+^2}{4\nu (t - s)} \right). \]  

(15)

Here \((\alpha)_+ = \max\{\alpha, 0\}\).

(4) Construct \(\omega_{B\nu}\) and \((w_{IB\nu}, w_{II\nu})\) by the iteration scheme and use the abstract Cauchy-Kowalewski theorem (ACK).

cf.) ACK; Nishida (’77), Kano-Nishida (’79), Safonov (’95)
4-13. Equation for $\omega_{B_v}$

\[
\begin{aligned}
\partial_t \omega_{B_v} - \nu \Delta \omega_{B_v} + J(\omega_E + \omega_{B_v}) \cdot \nabla \omega_{B_v} &= 0, \\
J(f) &= \nabla^\perp (-\Delta_D)^{-1} f, \\
\omega_{B_v}|_{t=0} &= 0,
\end{aligned}
\]

subject to the boundary condition

\[
\nu(\partial_2 \omega_{B_v} + (-\partial_1^2)^{\frac{1}{2}} \omega_{B_v}) = -\partial_2 (-\Delta_D)^{-1}(J(\omega_E + \omega_{B_v}) \cdot \nabla \omega_{B_v} + u_E \cdot \nabla \omega_E).
\]
4-14. Function space for $w_{B\nu}(t, x_1, X_2)$, $w_{IB\nu}(t, x_1, X_2)$, $w_{II\nu}(t, x)$

$$\varphi_{B\nu}^{(\mu, \rho)}(\xi_1, X_2) = \exp \left( \frac{(\mu - \nu^2 X_2)_+}{4} |\xi_1| + \rho X_2^2 \right),$$

$$\varphi_{II\nu}^{(\mu, \theta)}(\xi_1, x_2) = \exp \left( \frac{(\mu - x_2)_+}{4} |\xi_1| + \frac{\theta}{\nu} (6d_E - x_2)^2 \right).$$

$$\|f\|_{X_{B\nu}^{(\mu, \rho)}} = \|\varphi_{BL}^{(\mu, \rho)} \hat{f}(\xi_1, X_2)\|_{L_{\xi_1}^2 L_{X_2}^1},$$

$$\|f\|_{X_{II\nu}^{(\mu, \theta)}} = \|\varphi_{II\nu}^{(\mu, \theta)} \hat{f}(\xi_1, x_2)\|_{L_{\xi_1}^2 L_{X_2}^2}.$$ 

Here $\hat{f}(\xi_1, X_2) = \mathcal{F}_{x_1 \rightarrow \xi_1}[f(\cdot, X_2)](\xi_1)$. 
Lemma 2. Let $t > s \geq 0$, $\mu \geq 0$, $0 \leq \rho \leq 2^{-4}$, and $0 \leq \theta \leq 2^{-4}$. Then it follows that

\[
\|\varphi^{(\mu,\rho)}_{Bv} \mathcal{F} (R_{\nu} e^{\nu(t-s)\Delta_N} R_{\nu 1} f)\|_{L^2_{\xi_1} L^1_{x_2}} \leq C \|\varphi^{(\mu,\rho)}_{Bv} \mathcal{F} (f)\|_{L^2_{\xi_1} L^1_{x_2}},
\]

\[
\|\varphi^{(\mu,\theta)}_{IIv} \mathcal{F} (e^{\nu(t-s)\Delta_N} f)\|_{L^2_{\xi_1} L^2_{x_2}} \leq C \|\varphi^{(\mu,\theta)}_{Iv} \mathcal{F} (f)\|_{L^2_{\xi_1} L^2_{x_2}}.
\]
4-16. Equation for $\omega_{Iv}$

\[
\begin{align*}
\partial_t \omega_I - \nu \Delta \omega_I + u \cdot \nabla \omega_I + J(\omega_I) \cdot \nabla (\omega_E + \omega_{BL}) \\
= -J(\omega_{BL}) \cdot \nabla \omega_E + \nu \Delta \omega_E,
\end{align*}
\]

subject to the boundary condition

\[
\begin{align*}
u (\partial_2 \omega_I + (-\partial_1^2)^{\frac{1}{2}} \omega_I) \big|_{x_2=0} &= -J_1(u \cdot \nabla \omega_I) \big|_{x_2=0} \\
- J_1(J(\omega_I) \cdot \nabla (\omega_E + \omega_{BL})) \big|_{x_2=0} \\
- J_1(J(\omega_{BL}) \cdot \nabla \omega_E + \nu \Delta \omega_E) \big|_{x_2=0}.
\end{align*}
\]

(16)
4-17. Equation for $w_{II_v}$

\[
\begin{aligned}
\partial_t w_{II_v} - \nu \Delta w_{II_v} + u \cdot \nabla w_{II_v} &= -J(\omega_{I_v}) \cdot \nabla \omega_E + F_{II_v}, \\
u &= J(\omega_E + \omega_{B_v} + \omega_{I_v}), \\
F_{II_v} &= -J(\omega_{B_v}) \cdot \nabla \omega_E + \nu \Delta \omega_E,
\end{aligned}
\]

$$\partial_2 w_{II_v} \mid_{x_2=0} = 0,$$

$$w_{II_v} \mid_{t=0} = 0.$$

Remark. (1) $\omega = \omega_E + R_{\frac{1}{v}}w_{B_v} + \omega_{I_v}, \quad \omega_{I_v} = R_{\frac{1}{v}}w_{IB_v} + w_{II_v}$.

(2) supp $F_{II}(t) \subset \{ x \in \mathbb{R}_+^2 \mid x_2 \geq 32d_E > 0 \}$.

(3) Eq. for $w_{II_v}$ is the heat-convection equations with the homogeneous Neumann boundary condition.
4-18. Open problem (quite challenging)

1. The rigorous description of the behavior of vorticity fields when the instability of the boundary layer occurs.

2. For sufficiently general class of initial data the uniform bound of velocity fields such as

\[ \sup_{0<\nu\ll1} \sup_{0<t<T} \|u_{NS}^{(\nu)}(t)\|_{L^\infty} < \infty. \]

cf.) So far we have only \[ \sup_{0<\nu\ll1} \sup_{0<t<\nu^{3/2}} \|u_{NS}^{(\nu)}(t)\|_{L^\infty} < \infty. \]
5-1. Asymptotic expansion near the initial time

**Theorem 4.** Assume that $b \in W^{l,4/3}(\mathbb{R}^2_+)$ for $l \gg 1$ and satisfies the compatibility condition. Let $\omega$ be the solution to (V)-(BC). Then there are $c_0, C > 0$ such that the following estimates hold for sufficiently small $\nu > 0$:

\[
\|u(t)\|_{L^\infty} \leq C, \quad \|\omega(t) - \omega_E(t) - \omega_{BL}(t)\|_{L^p} \leq C \nu^{\frac{1}{2}\left(\frac{4}{3} - \frac{1}{p}\right)} t^{\frac{1}{2}(1 + \frac{1}{p})}
\]

(18)

for $0 < t \leq c_0 \nu^{1/3}$ and $4/3 \leq p \leq \infty$. Here $c_0$ is independent of $\nu$, and $C$ is independent of $\nu$ and $t \in [0, c_0 \nu^{1/3}]$.

The function $\omega_E$ is the vorticity field of the solution to the Euler equation with the initial velocity $a$. 
The function $\omega_{BL}$ describes the boundary layer and it is non-trivial if and only if

$$\partial_2 (-\Delta_D)^{-1}(a \cdot \nabla b) \neq 0 \quad \text{on} \quad \partial \mathbb{R}^2_+. \quad (19)$$

When (19) holds $\omega_{BL}$ satisfies

$$\|\omega_{BL}(t)\|_{L^p} \leq C' \nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})}$$

$$\|\omega_{BL}(t)\|_{L^p(\{0 \leq x_2 \leq (\nu t)^{\frac{1}{2}}\})} \geq c_1 \nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})}$$

for $t > 0$ and $1 \leq p \leq \infty$. Here the positive constants $c_1$ and $C'$ are independent of $\nu$ and $t.$
Corollary 1 (high vorticity creation near the initial time).

Under the assumptions of Theorem 4, if (19) holds in addition, then there is $c_2 > 0$ such that

$$
\|\omega(t) - \omega_E(t)\|_{L^p(\{0 \leq x_2 \leq (\nu t)^{1/2}\})} \geq c_2 \nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})}
$$

for $0 \leq t \leq c_0 \nu^{1/3}$ and $4/3 \leq p \leq \infty$. Here $c_0$ is the constant in Theorem 4 and $c_2$ is independent of $\nu$ and $t \in [0, c_0 \nu^{1/3}]$. In particular, high creation of vorticity near the boundary in $L^p$ occurs in the following sense:

$$
\|\omega(c_0 \nu^{1/3})\|_{L^p(\{0 \leq x_2 \leq c_0^{1/2} \nu^{3/2}\})} \geq c_3 \nu^{-\frac{1}{3}(1-\frac{2}{p})} \to \infty \quad (\nu \to 0)
$$

if $2 < p \leq \infty$. 
5-2. Remark on the vorticity creation. I

Corollary 1 shows the high creation of vorticity in $L^p$ near the boundary at $\nu \to 0$.

Although this is naturally expected from the boundary layer theory, the vorticity creation with explicit estimates has been obtained only under some restricted situations.

(i) Sammartino-Caflisch ('98); analytic initial data, creation in $L^p_{loc}$ for all $p > 1$.

(ii) Grenier ('00); around linearly-unstable stationary solutions to the Euler equations, creation in $L^\infty$. 
5-3. Remark on the vorticity creation. II

The condition (19) is necessary and sufficient for the vorticity to exhibit an unbounded growth at $T_\nu = c_0 \nu^{1/3}$ as $\nu \to 0$.

The meaning of (19) is nothing but $\partial_t u_{E,1}|_{t=0} \neq 0$ on $\partial \mathbb{R}_+^2$, where $u_E = (u_{E,1}, u_{E,2})$ denotes the solution to the Euler equations with the initial velocity $a$.

Hence (19) represents the nondegenerate condition for $u_E$ to be a nonzero velocity field on the boundary right after the initial time.

In such situations it is natural that the boundary layer immediately appears and thus the high vorticity creation occurs near the initial time.