Spacetime Quantum Geometry

Peter Schupp

Jacobs University Bremen

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Outline

Spacetime quantum geometry

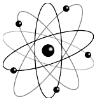
Applying the principles of quantum mechanics to space & time

Outline

- spacetime non-commutativity
- particle physics on non-commutative spaces
- non-commutative gravity and fuzzy black holes
- large scale application of small scale math (CMB analysis)
- higher geometric structures

Spacetime non-commutativity

Planck scale quantum geometry



Heuristic argument: quantum + gravity

"The gravitational field generated by the concentration of energy required to localize an event in spacetime should not be so strong as to hide the event itself to a distant observer."

$\rightarrow$ fundamental length scale, spacetime uncertainty

$$\Delta x \geq \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-33} \text{cm}$$

uncertainty principle $\leftrightarrow$ noncommutative spacetime structure

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}(x)$$

Spacetime non-commutativity

Macroscopic non-commutativity

- Landau levels
  charged particle in a constant magnetic field $\vec{B} = \nabla \times \vec{A}$:
  quantize $\vec{p} = m\dot{\vec{x}} - e\vec{A}$ conjugate to $\vec{x}$

  for $m \to 0$ (projection onto 1st Landau level):

$$[\hat{x}^i, \hat{x}^j] = \frac{2i}{eB}\epsilon^{ij} =: \theta^{ij}$$

- Fractional Quantum Hall Effect
  Non-commutative (NC) fluids, NC Chern Simons theory

Spacetime non-commutativity

Loop quantum gravity, spacetime foam

- non-perturbative framework
- defines quantum spacetime in algebraic terms
- integrating out gravitational degrees of freedom appears to yield effective NC actions with Lie-type NC structure

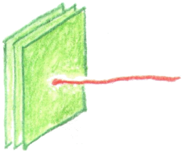
$$[\hat{x}^i, \hat{x}^j] = i\kappa\hbar\epsilon^{ijk}\hat{x}^k\,, \quad \kappa = 4\pi G \qquad (\kappa\text{-Minkowski space})$$

(feasible so far only for 2+1 dimensions)

Spacetime non-commutativity

Non-commutativity in string theory

- effective dynamics of open strings ending on D-branes: non-abelian gauge theory
- in closed string background B-field $\rightarrow$ non-commutative gauge theory



string endpoints become non-commutative:

$$\langle f_1(x(\tau_1)) \ldots f_n(x(\tau_n)) \rangle = \int dx \, f_1 \star \ldots \star f_n \,, \qquad \tau_1 < \ldots < \tau_n$$

with star product $\star$ depending on B.

Spacetime non-commutativity

Star product

General x-dependent NC structure:

$$f \star g = f \cdot g + \frac{i}{2} \sum \theta^{ij}\, \partial_i f \cdot \partial_j g - \frac{\hbar^2}{4} \sum \theta^{ij}\theta^{kl}\, \partial_i\partial_k f \cdot \partial_j\partial_l g$$
$$- \frac{\hbar^2}{6} \left(\sum \theta^{ij}\partial_j\theta^{kl}\, \partial_i\partial_k f \cdot \partial_l g - \partial_k f \cdot \partial_i\partial_l g \right) + \ldots$$

on coordinates:

$$[x^i \overset{\star}{,} x^j] = i\theta^{ij}$$

Spacetime non-commutativity

How does a quantum space look like?

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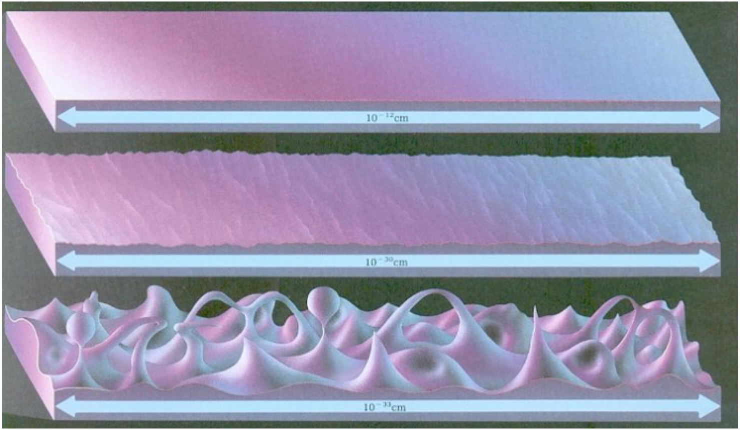

Spacetime non-commutativity

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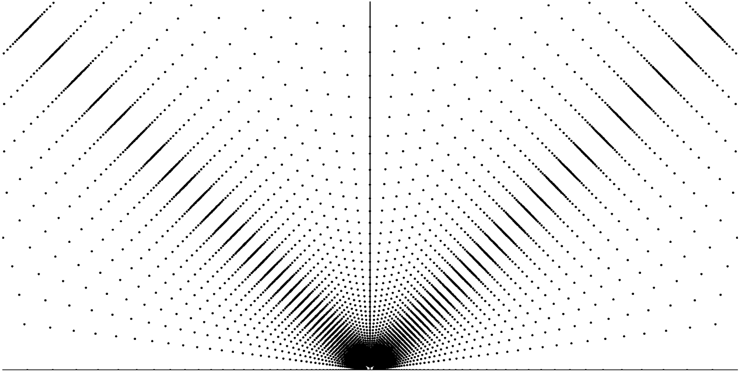


(q-Minkowski space; Cerchiai, Wess 1998)

Particle physics on non-commutative spaces

Quantum/Noncommutative Spacetime

- model of quantum geometry, spacetime uncertainty

Particle physics on non-commutative spaces

- construction of QFT on quantum spacetime
- experimental signatures?

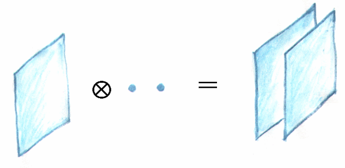
Features

- controlled Lorentz violation, non-locality, UV/IR mixing, mixing of internal & spacetime symmetries

Non-commutative Standard Model I

Connes, Lott; Madore (+ many others)

- spacetime augmented by a discrete space

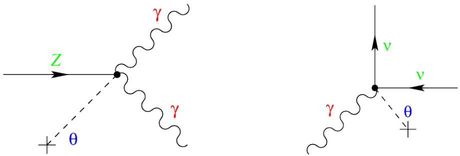


- Higgs field is NC gauge potential in the discrete direction
- beautiful geometrical interpretation of the full SM
- unlucky prediction of the Higgs mass

Non-commutative Standard Model II

Calmet, Jurco, PS, Wess, Wohlgenannt (+ many others)

- deformed spacetime with $\star$-product
- particle content and structure group of undeformed SM
- enveloping algebra formalism, SW maps
- many new SM forbidden interactions



Star product and Seiberg-Witten map

Star product:

$$f \star g = fg + \frac{1}{2}\theta^{\mu\nu}\partial_\mu f\,\partial_\nu g + \ldots .$$

Corresponding expansion of fields via SW map:

$$\begin{aligned}
\widehat{A}_\mu[A,\theta] &= A_\mu + \frac{1}{4}\theta^{\xi\nu}\{A_\nu,\partial_\xi A_\mu + F_{\xi\mu}\} + \ldots \\
\widehat{\Psi}[\Psi,A,\theta] &= \Psi + \frac{1}{2}\theta^{\mu\nu}A_\nu\partial_\mu\Psi + \frac{1}{4}\theta^{\mu\nu}\partial_\mu A_\nu\Psi + \ldots \\
\widehat{\Lambda}[\Lambda,A,\theta] &= \Lambda + \frac{1}{4}\theta^{\xi\nu}\{A_\nu,\partial_\xi\Lambda\} + \ldots
\end{aligned}$$

Deformed action

Matrix multiplication is augmented by $\star$-products.

e.g.: noncommutative Yang-Mills-Dirac action:

$$\widehat{S} = \int d^4x \left(-\frac{1}{4} Tr(\widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu}) + \overline{\widehat{\Psi}} \star i\widehat{\not{D}}\widehat{\Psi} \right)$$

with NC field strength

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i[\widehat{A}_\mu \stackrel{\star}{,} \widehat{A}_\nu]$$

Particle physics on non-commutative spaces

Non-perturbative effects at the quantum level

Beta function of U(N) NC Yang-Mills theory:

$$\beta(g^2) = \frac{\partial g^2}{\partial \ln \Lambda} = -\frac{22}{3}\frac{g^4 N^2}{8\pi^2}$$

like ordinary SU(N) gauge theory – but holds also for $N = 1$

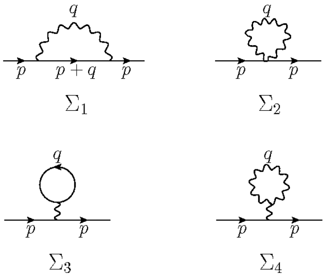
This U(1) NC gauge theory is asymptotically free with strong coupling effects at large distance scales.

Photon neutrino interaction

neutral particles can interact with photons via $\star$-commutator

$$D_\mu \Psi = \partial_\mu \Psi - i[A_\mu \stackrel{\star}{,} \Psi]$$

1-loop contributions to neutrino self-energy:



Particle physics on non-commutative spaces

Superluminal neutrinos?

deformed dispersion relation

$$p^2 \sim \underbrace{\left(\left(\frac{8\pi^2}{e^2} - 1 \right) \pm 2 \left(\frac{8\pi^2}{e^2} - 1 \right)^{\frac{1}{2}} \right)}_{\text{constant } k\text{, independent of } |\theta|} \cdot p_r^2 \,,$$

direction dependent neutrino velocity

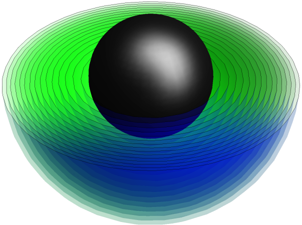
$$E^2 = |\vec{p}|^2 c^2 (1 + k \sin^2 \theta) \,.$$

Horvat, Ilakovac, PS, Trampetic, You: arXiv:1111.4951v1 [hep-th]

Non-commutative gravity

and exact solutions

- Gravity on non-commutative spacetime
- Fuzzy black hole solutions



Non-commutative gravity

Motivation

Quantum black holes

- nice theoretical laboratory for physics beyond QFT/GR
- information paradox, entropy, holography, singularities, . . .

Major obstacle

- The existence of a fundamental length scale is a priori incompatible with spacetime symmetries

$\Rightarrow$ The symmetry (Hopf algebra) must be deformed

Non-commutative gravity

Drinfel'd twisted symmetry:

$$\mathcal{F} = \exp\left(-\frac{i}{2}\theta^{ab}\, V_a \otimes V_b\right) \,, \qquad [V_a, V_b] = 0$$

$$\Delta_\star(f) = \mathcal{F}\Delta(f)\mathcal{F}^{-1} \qquad f \star g = \bar{\mathcal{F}}(f \otimes g)$$

Twisted tensor calculus

Two simple rules:

- The transformation of individual tensors is not deformed
- Tensors must be $\star$-multiplied

Twisted quantization, Hopf algebra symmetry in string theory:
Asakawa, Watamura (Tohoku University)
$\rightarrow$ afternoon session: T. Asakawa "NC solitions of gravity"

Non-commutative gravity

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Non-commutative gravity

Deformed Einstein equations

with “non-commutative sources” $T_{\mu\nu}$

$$R_{\mu\nu}(G,\star) = T_{\mu\nu} - \frac{1}{2}G_{\mu\nu}T$$

Solutions?

Solution = mutually compatible pair:

- algebra (twist) $\star$
- metric $G_{\mu\nu}$

Non-commutative black hole

Exact solution with rotational symmetry

Star product (twist) for tensors:

$$V \star W = VW + \sum_{n=1}^{\infty} C_n(\frac{\lambda}{\rho}) \mathcal{L}^n_{\xi_+} V \, \mathcal{L}^n_{\xi_-} W$$

left invariant Killing vector fields: $\xi_\pm = \xi_1 \pm i\xi_2$
commutative radius: $\rho^2 = x^2 + y^2 + z^2$

$$\begin{aligned} C_n(\frac{\lambda}{\rho}) &= B(n, \frac{\rho}{\lambda}) \\ &= \frac{\lambda^n}{n! \, \rho(\rho - \lambda)(\rho - 2\lambda) \cdots (\rho - (n-1)\lambda)} \end{aligned}$$

Non-commutative black hole

Metric

in isotropic coordinates

$$ds^2 = -\left(1 - \frac{a}{\rho}\right) dt^2 + \frac{r^2}{\rho^2}\left(dx^2 + dy^2 + dz^2\right)$$

with

$$r = (\rho + a/4)^2/\rho, \quad a = 2M$$

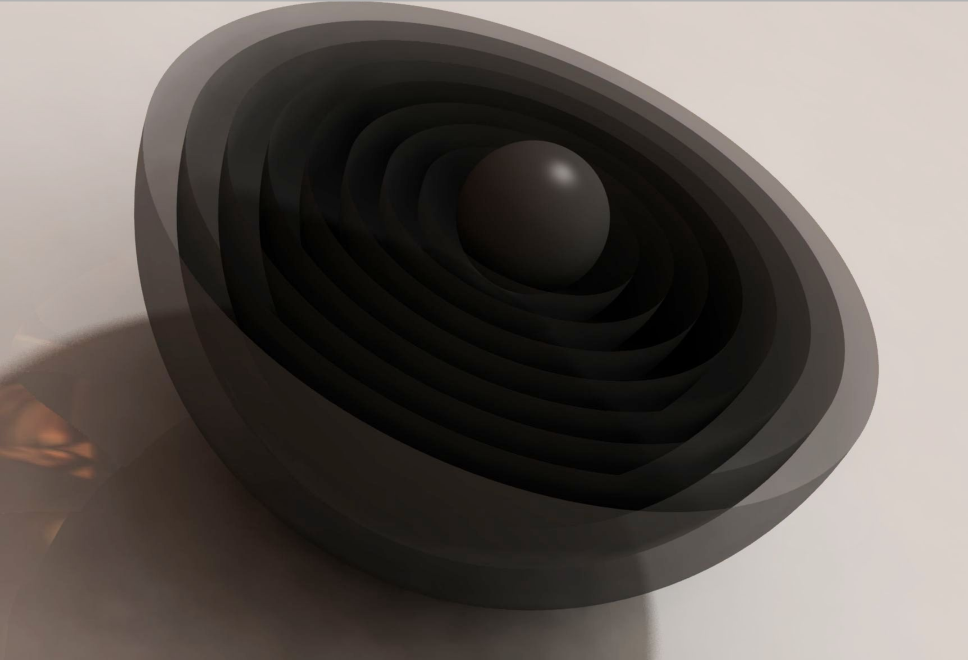
$$\rho^2 = g_{ij}x^i x^j = x^2 + y^2 + z^2$$

$$[x_i \overset{\star}{,} x_j] = 2i\lambda\epsilon_{ijk}x_k$$

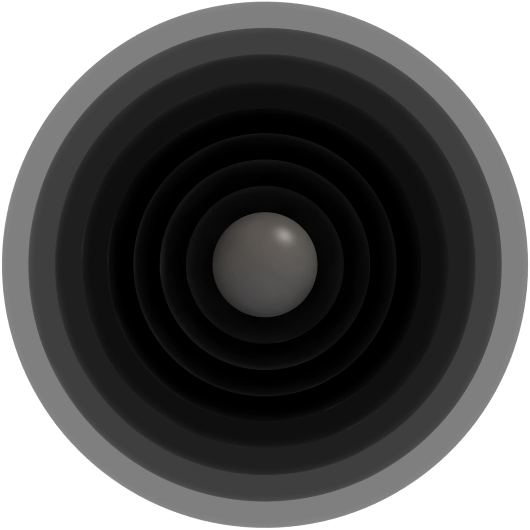
$\Rightarrow$ quantized, quasi 2-dimensional “onion”-spacetime:

$$\boxed{\rho = 2j\lambda = n\lambda; \quad n = 0, 1, 2, \ldots}$$

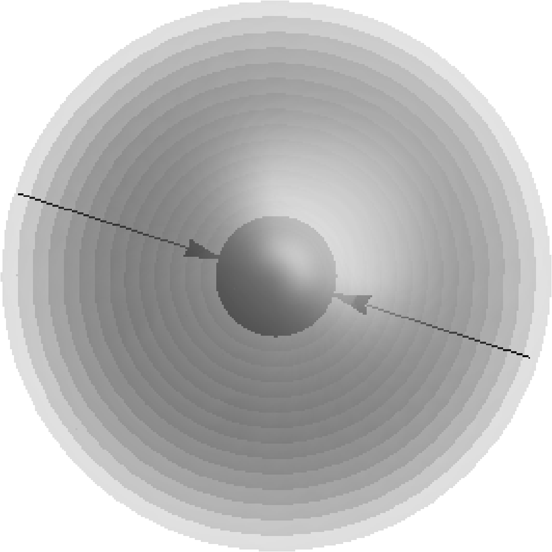
Non-commutative black hole



Non-commutative black hole



Non-commutative black hole



Non-commutative black hole

Hilbert Space

$L_2(\mathbb{R}^3) \to L_2(S_2)$

We find the following surprising result:
States describing events in 3 dim NC bulk are equivalent to wave functions on a sphere (minus a fuzzy sphere)

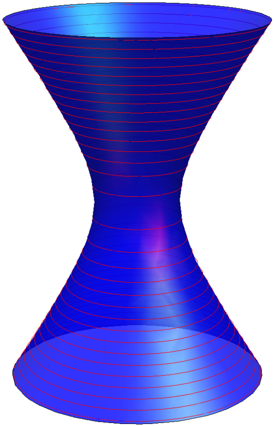
$$\mathcal{H} = \bigoplus_{n>N} \mathbb{C}^{n+1} = \bigoplus \mathbb{C}^{n+1} - \mathcal{H}_{\text{hidden}}$$

- NC Schwarzschild solution = "Fuzzy Black Hole"
- Holographic behavior appears quite naturally

NC: bulk (3D) $\to$ surface (2D)

Heuristically:
(1) coordinates are no longer independent: $z \sim [x, y]$
(2) number of commuting operators = two

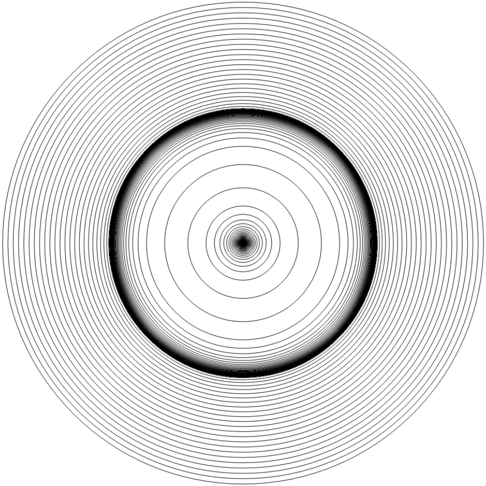
Inside NC black hole



two sequences of fuzzy spheres with limit points
at $r = 0$ (singularity) and $r = a$ (horizon)

Non-commutative black hole

Fuzzy black hole inside and outside, in one figure:



Coherent States, Star Products, Entropy

Generalized coherent state ($SU(2)$, spin j representation)

$$|\Omega\rangle = \mathcal{R}_\Omega |j, j\rangle, \quad \mathcal{R}_\Omega \in SU(2)/U(1); \qquad (2j+1)\int \frac{d\Omega}{4\pi} |\Omega\rangle\langle\Omega| = 1_j$$

Star product

For $A(\Omega) := \langle\Omega|A|\Omega\rangle$ and $B(\Omega) := \langle\Omega|B|\Omega\rangle$ define:

$$A(\Omega) \star B(\Omega) = \langle\Omega|AB|\Omega\rangle$$

Von Neumann entropy

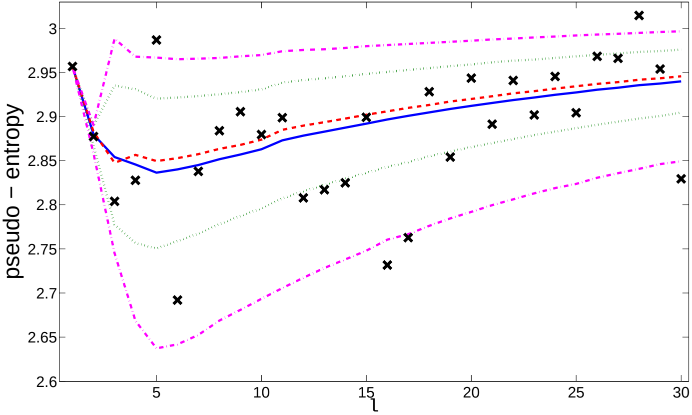
$$S_Q(\rho) = -\mathrm{tr}\rho \ln \rho = -(2j+1)\int \frac{d\Omega}{4\pi}\rho(\Omega) \star \ln_\star \rho(\Omega)$$

Now “switch off” (or ignore) noncommutativity $\Rightarrow$

Wehrl entropy

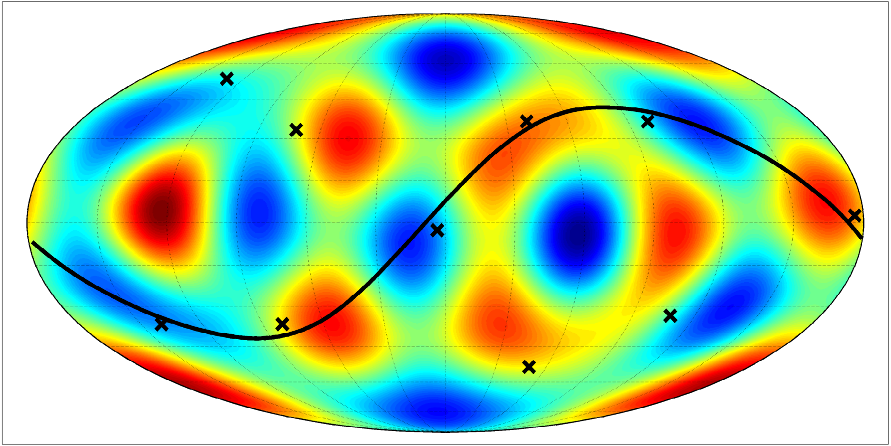
$$\begin{aligned} S_W(\rho) &= -(2j+1)\int \frac{d\Omega}{4\pi}\rho(\Omega) \ln \rho(\Omega) \\ &\geq -(2j+1)\int \frac{d\Omega}{4\pi}|\langle\Omega|\Psi\rangle|^2 \ln |\langle\Omega|\Psi\rangle|^2 \\ &> 0 \quad \text{even for pure states} \end{aligned}$$

Coherent state analysis of CMB



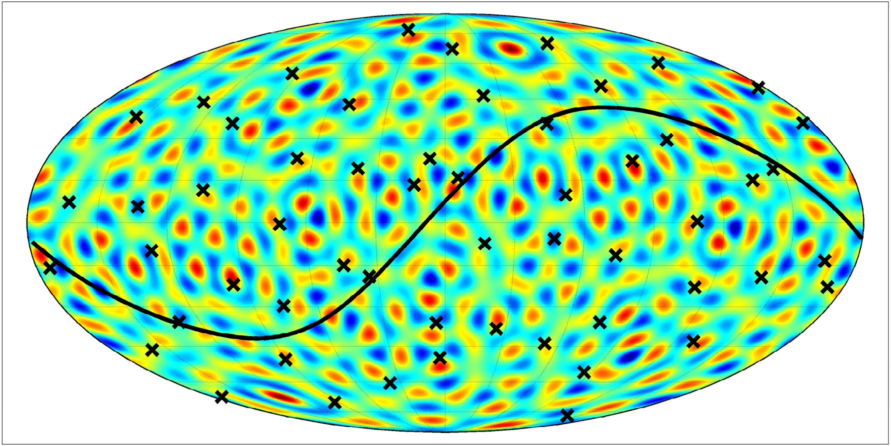
Coherent states and CMB

CMB at $j = 5$ with multi-pole vectors ($\sim$ coherent states)



Coherent states and CMB

CMB at $j = 28$



Higher geometric structures

Beyond "non-commutative"

Recent work in M-theory reveales non-associative higher geometric structures featuring Nambu brackets:

- Basu-Harvey equations, fuzzy S_3 funnels
- Bagger-Lambert action for multiple $M2$ and $M5$-branes

$\rightarrow$ afternoon session: M. Sato "3-algebra Model of M-Theory"

Higher geometric structures

Nambu mechanics

multi-Hamiltonian dynamics with generalized Poisson brackets

e.g. Euler's equations for the spinning top :

$$\frac{d}{dt} L_i = \{L_i, \frac{\vec{L}^2}{2}, T\} \qquad i = 1, 2, 3$$

with angular momenta L_1, L_2, L_2, kinetic energy $T = \sum \frac{L_i^2}{2I_i}$ and

Nambu-Poisson bracket

$$\{f, g, h\} \propto \det \left[\frac{\partial(f, g, h)}{\partial(L_1, L_2, L_3)} \right] = \epsilon^{ijk}\, \partial_i f\, \partial_j g\, \partial_k h$$

Higher geometric structures

Nambu-Poisson (NP) bracket

more generally:

$$\{f, h_1, \ldots, h_p\} = \Pi^{i\, j_1 \ldots j_p}(x)\, \partial_i f\, \partial_{j_1} h_1\, \cdots\, \partial_{j_p} h_p$$

+ Fundamental Identity (FI)

$$\begin{aligned}\{\{f_0, \cdots, f_p\}, h_1, \cdots, h_p\} &= \{\{f_0, h_1, \cdots, h_p\}, f_1, \cdots, f_p\} + \ldots \\ &\quad \ldots + \{f_0, \ldots, f_{p-1}, \{f_p, h_1, \cdots, h_p\}\}\end{aligned}$$

Higher geometric structures

Nambu-Dirac-Born-Infeld action

(B Jurco & PS 2012)

commutative $\leftrightarrow$ non-commutative symmetry implies

$$S_{DBI} = \int d^n x \frac{1}{g_m} \det{}^{\frac{p}{2(p+1)}} [g] \det{}^{\frac{1}{2(p+1)}} \left[g + (B+F)\tilde{g}^{-1}(B+F)^T \right]$$

$$= \int d^n x \frac{1}{G_m} \det{}^{\frac{p}{2(p+1)}} \left[\widehat{G} \right] \det{}^{\frac{1}{2(p+1)}} \left[\hat{G} + (\hat{\Phi} + \hat{F}) \widehat{\tilde{G}}^{-1} (\hat{\Phi} + \hat{F})^T \right]$$

This action interpolates between early proposals based on supersymmetry and more recent work featuring higher geometric structures.

Thank you for listening!