Toric Degeneration of Gelfand-Cetlin Systems and Potential Functions

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Completely integrable systems

Let (X, ω) be a symplectic manifold of dimension $2N$. A completely integrable system on (X, ω) is a set of N functions

$$\Phi = (f_1, \ldots, f_N) : X \longrightarrow \mathbb{R}^N$$

satisfying

- Poisson commutativity: $\{f_i, f_j\} = 0$ for $i, j = 1, \ldots, N$, and
- Functional independence.

If $\omega = \sum_i dp_i \wedge dq_i$, the Poisson bracket is given by

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$$

Arnold-Liouville Theorem: If fibers of Φ are compact, general fibers are (union of) Lagrangian tori:

$$\Phi^{-1}(p) = \text{(union of) } T^N,$$
$$\omega|_{\Phi^{-1}(p)} = 0.$$

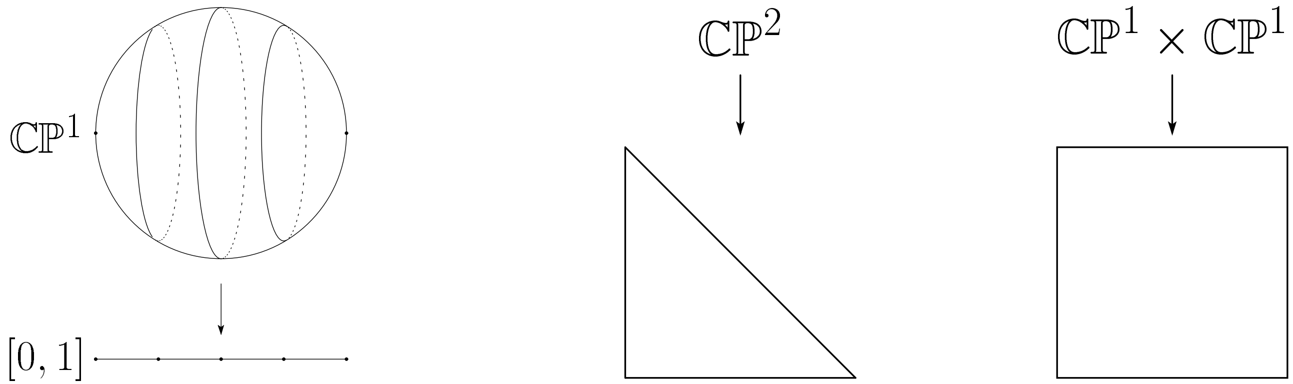
Example: Let X be a (compact) toric variety of $\dim_{\mathbb{C}} = N$ with a T^N-invariant Kähler form. Then the moment map

$$\Phi : X \to \mathbb{R}^N = (\mathrm{Lie}\, T^N)^*$$

of the T^N-action is a completely integrable system. $\Delta := \Phi(X) \subset \mathbb{R}^N$ is a convex polytope, called the moment polytope of X.

Example of Example: $X = \mathbb{CP}^1 \cong S^2$.

$$\Phi : \mathbb{CP}^1 \longrightarrow \mathbb{R}, \qquad [z_0 : z_1] \longmapsto \frac{|z_1|^2}{|z_0|^2 + |z_1|^2}$$



Remark. In general, if $p \in \Delta$ is a point in a k-dimensional face, then

$$\Phi^{-1}(p) \cong T^k.$$

Flag manifolds

Flag manifold is a complex manifold defined by

$$\begin{aligned} Fl_n &:= \{0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \,|\, \dim V_i = i\} \\ &= U(n)/T, \end{aligned}$$

where $T \subset U(n)$ is a maximal torus. Fixing

$$\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \qquad \lambda_1 > \lambda_2 > \cdots > \lambda_n,$$

Fl_n is identified with the adjoint orbit of λ:

$$\begin{aligned} &Fl_n \cong \mathcal{O}_\lambda = \left\{ x \in M_n(\mathbb{C}) \;\middle|\; x^* = x, \text{ eigenvalues} = \lambda_1, \ldots, \lambda_n \right\} \\ &[g] \leftrightarrow g\lambda g^* \end{aligned}$$

$\mathcal{O}_\lambda$ has a standard symplectic form:

$$\omega_\lambda := \text{Kostant-Kirillov form (a } U(n)\text{-invariant Kähler form).}$$

Gelfand-Cetlin systems

For each $x \in \mathcal{O}_\lambda$, set

$$x^{(k)} = \text{upper-left } k \times k \text{ submatrix of } x,$$
$$\lambda_1^{(k)}(x) \geq \cdots \geq \lambda_k^{(k)}(x) \ : \text{ eigenvalues of } x^{(k)}.$$

Theorem (Guillemin-Sternberg).

$$\Phi_\lambda : \mathcal{O}_\lambda \longrightarrow \mathbb{R}^{n(n-1)/2}, \quad x \longmapsto \left(\lambda_i^{(k)}(x)\right)_{\substack{k=1,\ldots,n-1,\\ i=1,\ldots,k}}$$

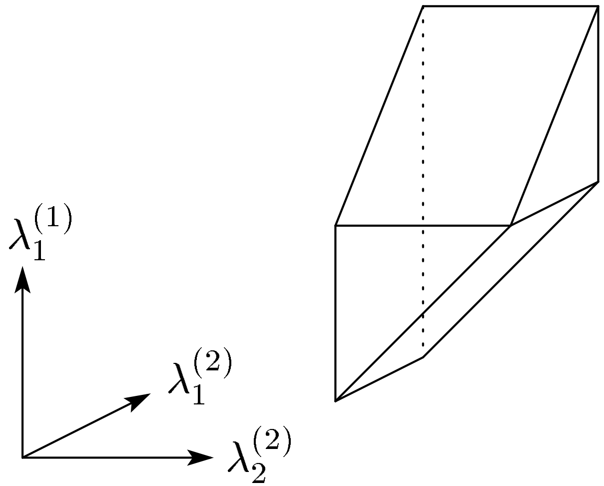
is a completely integrable system on (Fl_n, ω_λ).

Φ_λ is called the Gelfand-Cetlin system. The image $\Delta_\lambda = \Phi_\lambda(\mathcal{O}_\lambda)$ is a convex polytope, called the Gelfand-Cetlin polytope.

Example (the case of Fl_3).

$$\Phi_\lambda = (\lambda_1^{(2)}, \lambda_2^{(2)}, \lambda_1^{(1)}) : Fl_3 \longrightarrow \mathbb{R}^3.$$

Gelfand-Cetlin polytope Δ_λ is:



Every fiber of an interior point is a Lagrangian T^3.
The fiber of the vertex emanating four edges is a Lagrangian S^3.

Toric degeneration of flag manifolds

There exists a (singular) toric variety whose moment polytope is Δ_λ. We call this the Gelfand-Cetlin toric variety.

Theorem (Gonciulea-Lakshmibai, ...). *There exists a flat family*

$$f : \mathfrak{X} \longrightarrow S$$

of projective varieties such that $X_1 = f^{-1}(s_1)$ *is* Fl_n *and a special fiber* $X_0 = f^{-1}(s_0)$ *is the Gelfand-Cetlin toric variety.*

Toric degeneration is given by deforming the Plücker embedding

$$Fl_n \hookrightarrow \prod_{i=1}^{n-1} \mathbb{P}\Big(\bigwedge^{i} \mathbb{C}^n\Big), \quad (V_1 \subset \cdots \subset V_{n-1}) \mapsto (\wedge^1 V_1, \ldots, \wedge^{n-1} V_{n-1}).$$

Example (Toric degeneration of Fl_3).

$$Fl_3 = \left\{ \Big([z_0 : z_1 : z_2], [w_0 : w_1 : w_2] \Big) \in \mathbb{P}^2 \times \mathbb{P}^2 \;\Big|\; z_0 w_0 = z_1 w_1 + z_2 w_2 \right\}.$$

Its toric degeneration is given by

$$\mathfrak{X} = \left\{ \Big([z_0 : z_1 : z_2], [w_0 : w_1 : w_2], \boldsymbol{t} \Big) \;\Big|\; \boldsymbol{t} z_0 w_0 = z_1 w_1 + z_2 w_2 \right\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{C}$$

$X_1 = \left\{ z_1 w_1 + z_2 w_2 = z_0 w_0 \right\}$ Flag manifold,

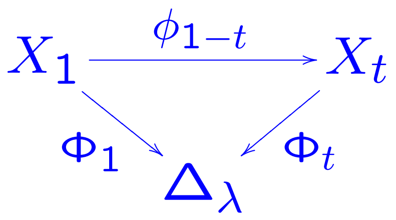
$X_0 = \left\{ z_1 w_1 + z_2 w_2 = 0 \right\}$ Gelfand-Cetlin toric variety.

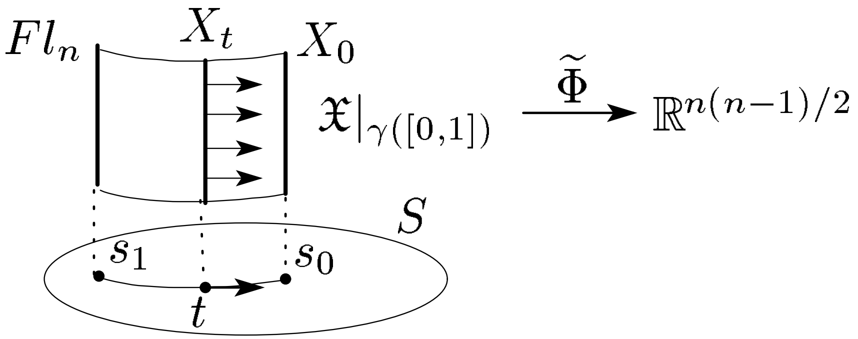
Toric degeneration of Gelfand-Cetlin systems

The Gelfand-Cetlin system can be deformed into the moment map on the Gelfand-Cetlin toric variety in the following sense:

Theorem. *There exist a path* $\gamma : [0,1] \to S$ *with* $\gamma(0) = s_0$, $\gamma(1) = s_1$, *a map* $\widetilde{\Phi} : \mathfrak{X}|_{\gamma([0,1])} \to \mathbb{R}^{n(n-1)/2}$, *and a flow* $\phi_t : X_1 \to X_{1-t}$ *s.t.*

- $\Phi_0 := \widetilde{\Phi}|_{X_0}$ *is the moment map on* $X_0 = f^{-1}(\gamma(0))$,
- Φ_1 *is the Gelfand-Cetlin system on* $X_1 = Fl_n$,
- Φ_t *is a completely integrable system on* X_t *for each* t,
- ϕ_t *preserves the structure of completely integrable systems:*





Application to mirror symmetry

Mirror symmetry is a duality in string theory.
Mathematically: duality between symplectic geometry on X and complex geometry on Y, and vice versa.

Mirror of a Fano manifold X: Landau-Ginzburg model $(Y, \mathcal{F})$

- Y is a non-compact complex manifold,
- $\mathcal{F} : Y \longrightarrow \mathbb{C}$ is a holomorphic function (superpotential).

Example. Mirror of $\mathbb{CP}^1$ is given by

$$Y \cong \mathbb{C}^*, \quad \mathcal{F}(y) = y + \frac{Q}{y},$$

where Q is a parameter.

Question: How to construct a L-G mirror $(Y, \mathcal{F})$ geometrically for a given Fano manifold X?

Potential Function (Fukaya-Oh-Ohta-Ono)

Roughly speaking, potential function $\mathfrak{PO}$ is a function on the space of Lagrangian submanifolds given by

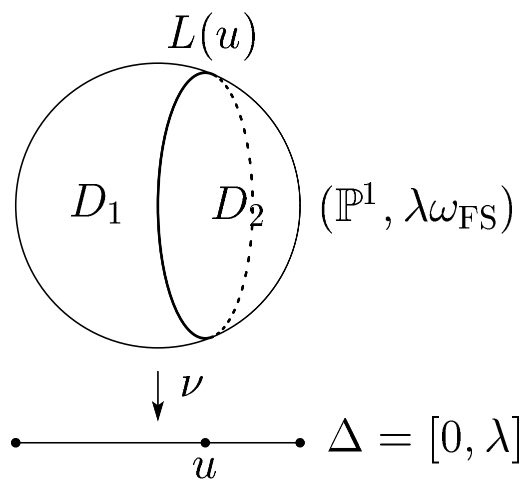
$$\mathfrak{PO}(L) = \sum_{\substack{\phi : D^2 \to X \text{ holo.}, \\ \phi(\partial D^2) \subset L}} e^{-\mathrm{Area}(\phi(D^2))},$$

where $\mathrm{Area}(\phi(D^2)) = \int_{D^2} \phi^* \omega$ is the symplectic area of $\phi(D^2)$.

Example ($\mathbb{CP}^1$ case).

$$\begin{aligned} \mathfrak{PO}(L(u)) &= e^{-\mathrm{Area}(D_1)} + e^{-\mathrm{Area}(D_2)} \\ &= e^{-u} + e^{-\lambda + u} \\ &= y + \frac{Q}{y}, \end{aligned}$$

where $y = e^{-u}$, $Q = e^{-\lambda}$.



Toric Fano case:

Theorem (Cho-Oh, Fukaya-Oh-Ohta-Ono)**.** *For a smooth toric Fano manifold X, the potential function $\mathfrak{PO}$ for Lagrangian torus fibers of the moment map is calculated explicitly from combinatorial data of the moment polytope. Moreover, $\mathfrak{PO}$ gives the superpotential of the Landau-Ginzburg mirror of X.*

Flag case:

Using toric degeneration of the Gelfand-Cetlin system, we have:

Theorem. *The potential function $\mathfrak{PO}$ for Lagrangian torus fibers of the Gelfand-Cetlin system is also calculated from Δ_λ, and gives the Givental's superpotential of the mirror of Fl_n.*

More precisely: Suppose that the Gelfand-Cetlin polytope Δ_λ is given by linear inequalities $\ell_i(u) \geq 0$. Then the potential function $\mathfrak{PO}$: $\mathrm{Int}\,\Delta_\lambda \to \mathbb{C}$ is given by

$$\mathfrak{PO}(u) = \sum_i e^{-\ell_i(u)}.$$

Example (The case of Fl_3).

$$\begin{aligned}\mathfrak{PO} &= e^{u_1-\lambda_1} + e^{-u_1+\lambda_2} + e^{u_2-\lambda_2} + e^{-u_2+\lambda_3} + e^{-u_1+u_3} + e^{u_2-u_3} \\ &= \frac{Q_1}{y_1} + \frac{y_1}{Q_2} + \frac{Q_2}{y_2} + \frac{y_2}{Q_3} + \frac{y_1}{y_3} + \frac{y_3}{y_2},\end{aligned}$$

where $y_k = e^{-u_k}$ and $Q_j = e^{-\lambda_j}$.