Asymptotic Stability of Stationary Wave for Damped Wave Equation with Non-Convex Convection Term

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1. Introduction

Half space : $(t, x) \in (0, \infty) \times \mathbb{R}_+$

- Damped wave equation with a nonlinear convection term

$$\begin{cases} u_{tt} - u_{xx} + u_t + f(u)_x = 0, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \\ u(t, 0) = u_b. \end{cases} \tag{1}$$

where

$u : (0, \infty) \times \mathbb{R}_+ \longrightarrow \mathbb{R}$; unknown function,

$f : \mathbb{R} \longrightarrow \mathbb{R}$; given smooth function with $f(0) = 0$.

Assumption : $u_0(x) \to 0 \quad (x \to \infty), \qquad u_b < 0$

Our Aim

Derive the asymptotic stability of the corresponding stationary wave for the damped wave equation (1) with non-convex convection term

Condition of flux $f(u)$

- $f''(u) > 0, \quad |f'(u)| < 1 \quad \text{for} \quad u \in [u_b, 0]$

$\Longrightarrow$ Obtained asymptotic stability and convergence rate. (U '08)

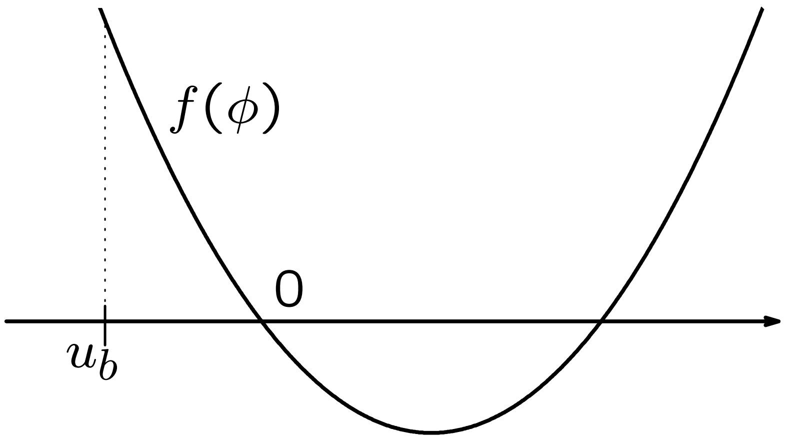
- $f''(0) > 0, \quad |f'(0)| < 1, \quad f(u) > 0 \quad \text{for} \quad u \in [u_b, 0] \qquad (2)$

$\Longrightarrow$ Obtained asymptotic stability for viscous conservation laws.

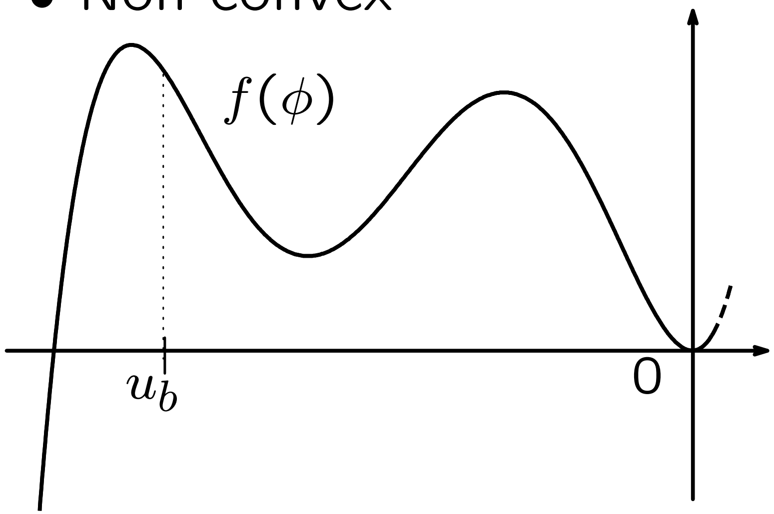
($u_t + f(u)_x = u_{xx}$, Hashimoto-Matsumura '08)

We get the asymptotic stability and the convergence rate for problem (1) under the condition (2).

- Convex



- Non-convex



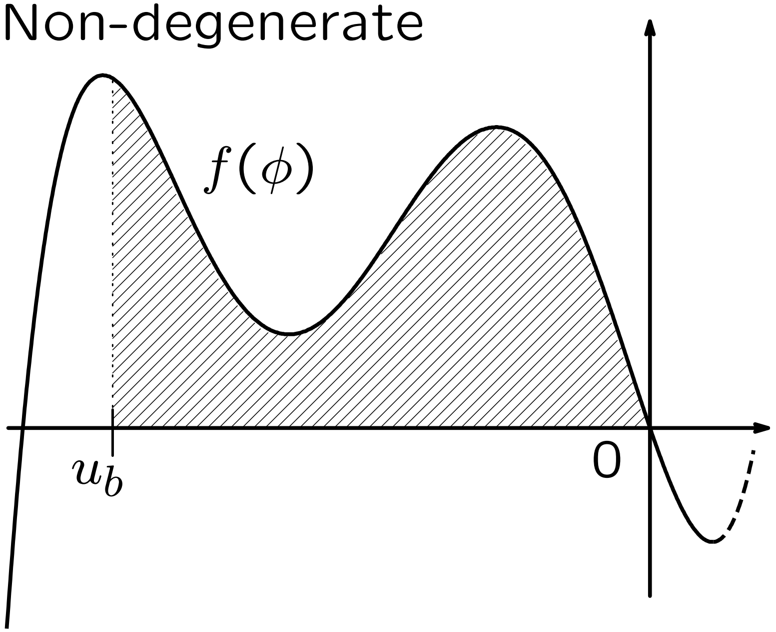
Stationary wave $\phi(x)$

Stationary problem : $$\begin{cases} -\phi_{xx} + f(\phi)_x = 0, \\ \phi(0) = u_b, \qquad \lim_{x \to \infty} \phi(x) = 0. \end{cases} \tag{3}$$

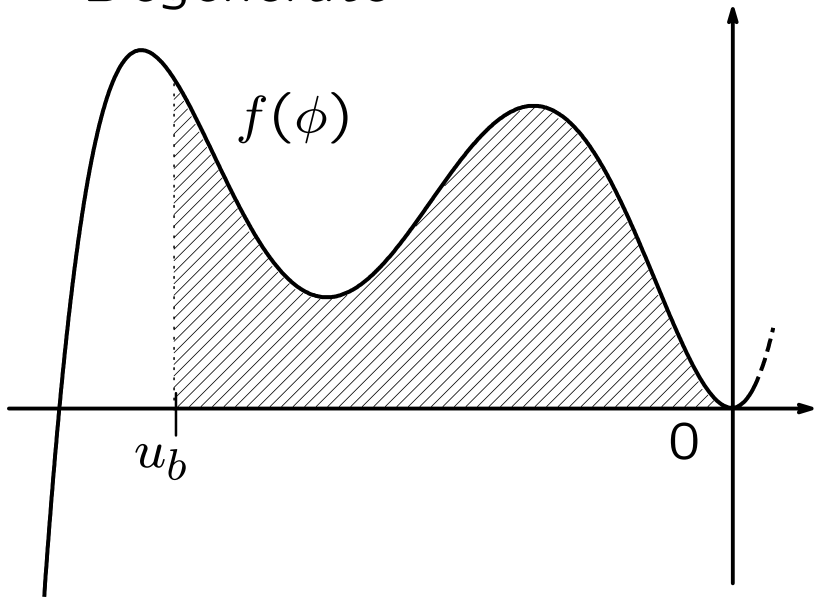
$$\Downarrow$$

$$f(\phi) = \phi_x$$

- Non-degenerate



- Degenerate



Lemma 1. (Hashimoto-Matsumura '08)

- $f'(0) \le 0 \iff$ (3) has a unique solution ϕ

(i) In the Non-degenerate case $f'(0) < 0$:

$$u_b < u_+ (= 0) \implies \phi(x) \text{ is monotone increasing,}$$

$$|\partial_x^k(\phi(x) - u_+)| \le C\delta e^{-cx} \qquad (\delta := |u_b|,\ k \ge 0).$$

(ii) In the Degenerate case $f'(0) = 0$:

$$u_b < u_+ (= 0) \implies \phi(x) \text{ is monotone increasing.}$$

$$|\partial_x^k(\phi(x) - u_+)| \le \frac{C\delta^{k+1}}{(1 + \delta x)^{k+1}}$$

Known result

Viscous conservation law $(u_t + f(u)_x = u_{xx})$

Liu-Matsumura-Nishihara '98 $\cdots$ $\|u-\phi\|_{L^\infty} \to 0 \quad as \quad t \to \infty$
(ϕ : Stationary wave)

Kawashima-Nishibata-Nishikawa '03 $\cdots$ $\|u-\phi\|_{L^\infty} \le C(1+t)^{-\alpha/2}$
(ϕ : Non-degenerate stationary wave)

Kawashima-Nakamura-U '07 $\cdots$ $\|u-\phi\|_{L^\infty} \le C(1+t)^{-\alpha/4}$
(ϕ : Degenerate stationary wave)

Hashimoto-Matsumura '08 $\cdots$ $\|u-\phi\|_{L^\infty} \to 0 \quad as \quad t \to \infty$
(ϕ : Stationary wave, f : Non-convex)

Damped wave eq. with nonlinear convection

U '06 $\cdots\cdots\cdots\cdots$

$\|u-\phi\|_{L^\infty} \to 0 \quad as \quad t \to \infty$ (ϕ : Stationary wave)

$\|u-\phi\|_{L^\infty} \le C(1+t)^{-\alpha/2}$ (ϕ : Non-degenerate stationary wave)

Kawashima-Nakamura-U '07 $\cdots$ $\|u-\phi\|_{L^\infty} \le C(1+t)^{-\alpha/4}$
(ϕ : Degenerate stationary wave)

Notation

$$L^2(\mathbb{R}_+) := \left\{ u \;\middle|\; \|u\|_{L^2} := \int_0^\infty |u(x)|^2 dx < \infty \right\},$$

$$H^1(\mathbb{R}_+) := \{ u \in L^2(\mathbb{R}_+) \mid u_x \in L^2(\mathbb{R}_+) \},$$

Non-convex condition of flux $f(u)$

- $f''(0) > 0, \quad |f'(0)| < 1, \quad f(u) > 0 \quad$ for $\quad u \in [u_b, 0]$

Theorem 2. (Asymptotic stability)

Assume the non-convex condition. $\phi(x)$: Stationary wave.

$u_0 - \phi \in H^1, \quad u_1 \in L^2, \quad E_0 := \|u_0 - \phi\|_{H^1} + \|u_1\|_{L^2} \ll 1.$

$\Longrightarrow \exists 1 u(t,x)$: Time global solution of (1) s.t.

$$u - \phi \in C\Big([0,\infty); H^1(\mathbb{R}_+)\Big) \cap C^1\Big([0,\infty); L^2(\mathbb{R}_+)\Big).$$

$$\|u(t) - \phi\|_{L^\infty} \longrightarrow 0, \qquad t \to \infty.$$

Note that Theorem 2 includes the known result which derived by U'08.

Notation

$$L^2_\alpha(\mathbb{R}_+) := \left\{ u \;\middle|\; \|u\|_{L^2_\alpha} := \int_0^\infty (1+|x|)^\alpha |u(x)|^2 dx < \infty \right\},$$

$$H^1_\alpha(\mathbb{R}_+) := \{u \in L^2_\alpha(\mathbb{R}_+) \mid u_x \in L^2_\alpha(\mathbb{R}_+)\},$$

$$L^{2,\alpha}(\mathbb{R}_+) := \left\{ u \;\middle|\; \|u\|_{L^{2,\alpha}} := \int_0^\infty e^{\alpha x} |u(x)|^2 dx < \infty \right\},$$

$$H^{1,\alpha}(\mathbb{R}_+) := \{u \in L^{2,\alpha}(\mathbb{R}_+) \mid u_x \in L^{2,\alpha}(\mathbb{R}_+)\}.$$

Theorem 3. (Algebraic decay rate)

$f'(0) < 0$, $u(t,x)$: Global solution of (3), $\phi(x)$: Stationary wave

$$E_\alpha := \|u_0 - \phi\|_{H^1_\alpha} + \|u_1\|_{L^2_\alpha} \quad (\alpha > 0), \qquad E_0 \ll 1$$

$$\Longrightarrow \quad \|u(t) - \phi\|_{H^1} \le C E_\alpha (1+t)^{-\alpha/2}.$$

Theorem 4. (Exponential decay rate)

$$f'(0) < 0, \quad \tilde{E}_\beta := \|u_0 - \phi\|_{H^{1,\beta}} + \|u_1\|_{L^{2,\beta}} \quad (\beta > 0), \quad E_0 \ll 1$$

$$\Longrightarrow \quad \exists \alpha > 0 \quad s.t. \quad \|u(t) - \phi\|_{H^1} \le C \tilde{E}_\beta e^{-\alpha t}.$$

Outline of the proof of Theorem 2.

- Characteristic curve, Successive approximation

$\Longrightarrow$ Existence of the local solution

- Weighted L^2 energy method (c.f. Hashimoto-Matsumura '08)

$\Longrightarrow$ *A priori* estimate

$v(t,x) := u(t,x) - \phi(x)$

$$\Longrightarrow \begin{cases} v_{tt} - v_{xx} + v_t + \Big(f(\phi+v) - f(\phi)\Big)_x = 0, \\ v(0,x) = v_0(x) := u_0(x) - \phi(x), \quad v_t(0,x) = v_1(x) := u_1(x), \\ v(t,0) = 0. \end{cases}$$

$v(t,x) := w(\phi)\tilde{v}(t,x), \quad w$: Weight (c.f. Mei-Matsumura '97)

$$\Longrightarrow \begin{cases} w\tilde{v}_{tt} - (w\tilde{v})_{xx} + w\tilde{v}_t + \Big(f(\phi+w\tilde{v}) - f(\phi)\Big)_x = 0, \\ \tilde{v}(0,x) = \tilde{v}_0(x) := v_0/w, \quad \tilde{v}_t(0,x) = \tilde{v}_1(x) := v_1/w, \\ \tilde{v}(t,0) = 0. \end{cases} \tag{4}$$

$2\tilde{v}_t \times (4) + \tilde{v} \times (4) \implies$

$$(E + D_2)_t + D_1 + \frac{1}{2} D_2 + (F_1 - F_2)_x = 0. \qquad (5)$$

where

$$E := w\tilde{v}_t^2 + w\tilde{v}_x^2 + \frac{1}{2} w\tilde{v}^2 + w\tilde{v}_t\,\tilde{v},$$

$$D_1 := w\tilde{v}_t^2 + w\tilde{v}_x^2 + 2\Big(f'(\phi + v)w - w_x\Big)\tilde{v}_t\tilde{v}_x,$$

$$D_2 := -w_{xx}\tilde{v}^2 + 2\phi_x \int_0^{\tilde{v}} \Big(f'(\phi + w\eta) - f'(\phi)\Big) d\eta + 2\phi_x \int_0^{\tilde{v}} f'(\phi + w\eta) w' \eta d\eta,$$

$$F_1 := \Big(f(\phi + w\tilde{v}) - f(\phi)\Big)\tilde{v} - \int_0^{\tilde{v}} \Big(f(\phi + w\eta) - f(\phi)\Big) d\eta - \frac{1}{2} w_x \tilde{v}^2,$$

$$F_2 := 2w\tilde{v}_t\tilde{v}_x + \tilde{v}_x\tilde{v}.$$

By the simple calculation, we have

$$D_1 = w\tilde{v}_t^2 + w\tilde{v}_x^2 + 2(f'w - w'f)\tilde{v}_t\tilde{v}_x + \tilde{v}_t\tilde{v}_x O(|\tilde{v}|),$$

$$D_2 = \phi_x (f''w - w''f)\tilde{v}^2 + \phi_x O(|\tilde{v}|^3)$$

In order to get the *a priori* estimate, we have to make a positivity for D_1 and D_2. $\implies$ Choose the weight function w !!

Weight function.

$$f''(0) > 0 \iff \exists \nu, \ \exists r \quad s.t. \quad f''(u) \geq \nu \quad for \quad u \in [-r, r]$$

Lemma 5.

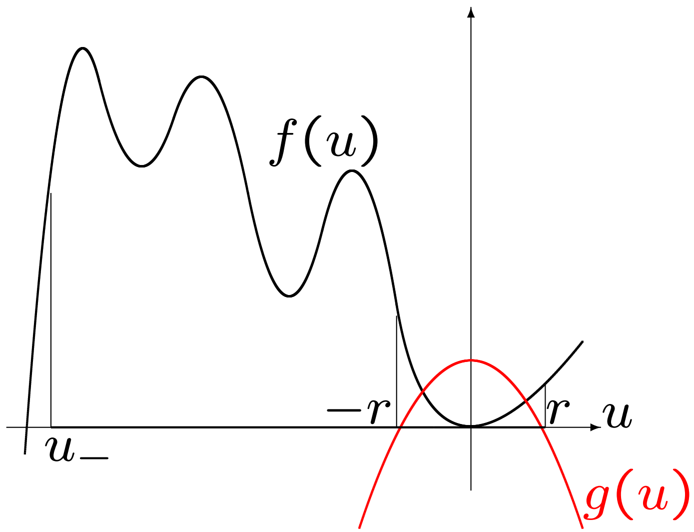
Put

$$w(\phi) = f(\phi) + \delta g(\phi), \quad g(u) := -u^{2m} + r^{2m}$$

Then, there exists δ and m s.t.

$$\text{(i) } c \leq w \leq C, \quad \text{(ii) } f''w - w''f \geq c, \quad \text{(iii) } (f'w - w'f)^2 < w^2$$

for $\phi \in [u_b, 0]$, where c and C are positive constants.



By using Lemma 5, we have

$$D_1 \geq c\,(\tilde{v}_t^2 + \tilde{v}_x^2), \quad D_2 \geq c\,\phi_x\,\tilde{v}^2 \qquad \text{for} \quad \|v\|_{L^\infty} \ll 1.$$

Thus, integrating (5) on $(0,t) \times \mathbb{R}_+$, we obtain the following *a priori* estimate :

$$\|(v_t, v_x, v)(t)\|_{L^2}^2 + \int_0^t \|(v_t, v_x)(\tau)\|_{L^2}^2 d\tau + \int_0^t \int_0^\infty \phi_x v^2 dx d\tau \leq CE_0.$$

(Here we use the simple relation $E \sim v^2 + v_x^2 + v_t^2$.)

Outline of the proof of Theorem 3

$(1+x)^\beta \times (4)$ $(\beta \in \mathbb{R})$ $\Longrightarrow$

$$\begin{aligned} &(1+x)^\beta (E+D_2)_t + (1+x)^\beta \Big(D_1 + \frac{1}{2} D_2\Big) \\ &\qquad - \beta (1+x)^{\beta-1} (F_1 - F_2) + \Big\{ (1+x)^\beta (F_1 - F_2) \Big\}_x = 0. \end{aligned} \tag{6}$$

Moreover, in the Non-degenerate case: $f'(0) < 0$, we obtain that

$$-F_1 = \frac{1}{2}(w'f - f'w)\tilde{v}^2 + O(|\tilde{v}|^3) \geq c\tilde{v}^2 \quad \text{for} \quad \|v\|_{L^\infty} \ll 1$$

Lemma 6.

Assume $f'(0) < 0$ (Non-degenerate case), then we have

$$w'f - f'w \geq c,$$

where c is positive constant.

Therefore, multiplying $(1+t)^\gamma$ and integrating (5) on $(0,t) \times \mathbb{R}_+$ and applying the induction argument, we have the desired estimate.

Weighted energy method.

For $\beta, \gamma \in \mathbb{R}$, we obtain

$$
\begin{aligned}
&(1+t)^{\gamma}\|(v_t,\, v_x,\, v)(t)\|_{L^2_\beta}^2 + \int_0^t (1+\tau)^{\gamma}\|(v_t,\, v_x)(\tau)\|_{L^2_\beta}^2 d\tau \\
&\qquad + \int_0^t (1+\tau)^{\gamma} \int_0^{\infty} (1+x)^{\beta}\phi_x v^2 dx d\tau + \beta \int_0^t (1+\tau)^{\gamma}\|v\|_{L^2_{\beta-1}}^2 d\tau \\
&\quad \le C E_\beta^2 + \gamma\, C \int_0^t (1+\tau)^{\gamma-1}\|(v_t,\, v_x,\, v)(\tau)\|_{L^2_\beta}^2 d\tau \\
&\qquad\qquad + \beta\, C \int_0^t (1+\tau)^{\gamma}\|(v_t,\, v_x)(\tau)\|_{L^2_{\beta-1}}^2 d\tau,
\end{aligned} \tag{7}
$$

where $E_\beta := \|u_0 - \phi\|_{H^1_\beta} + \|u_1\|_{L^2_\beta}$.

Put $\beta = 1$, $\gamma = 0$ in (7) $\Longrightarrow$

$$
\|(v_t,\, v_x,\, v)(t)\|_{L^2_1}^2 + \int_0^t \|(v_t,\, v_x)(\tau)\|_{L^2_1}^2 d\tau + \int_0^t \|v\|_{L^2}^2 d\tau \le C E_1^2 \tag{8}
$$

Put $\beta = 0$, $\gamma = 1$ in (7) + (8) $\Longrightarrow$

$$
(1+t)\|(v_t,\, v_x,\, v)(t)\|_{L^2}^2 \le C E^2 + C \int_0^t \|(v_t,\, v_x,\, v)(\tau)\|_{L^2}^2 d\tau \le C E_1^2
$$

Lemma 5.

(i) $c \leq w \leq C$, (ii) $f''w - w''f \geq c$, (iii) $(f'w - w'f)^2 < w^2$.

Outline of the Proof of the inequality (ii) in Lemma 5.

$$f''w - fw'' = \delta(f''g - fg''), \qquad \left(w(u) := f(u) - \delta g(u), \quad g(u) := -u^{2m} + r^{2m}\right)$$

1. $f''g > 0$, $-fg'' \geq 0$ for $u \in (-r, r)$.

$\Longrightarrow$ $f''g - fg'' > 0$ for $u \in (-r, 0]$.

2. $-fg'' > 0$ for $u \in [u_b, 0)$.

$\Longrightarrow$ $\exists m$ $(\gg 1)$ s.t. $f''g - fg'' > 0$

for $u \in [u_b, -r]$.