The 1st GCOE International Symposium

"Weaving Science Web beyond Particle-Matter Hierarchy"

Emergence and Annihilation of Vortices for Ginzburg-Landau Equations

Fukushima University

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Multimedia Education and Research Complex

(Kawauchi Campus of Tohoku University), Sendai

Plan of This Talk

- Ginzburg-Landau Equation
- Motivation (Emergence and Annihilation)
- On vortex solution
- Some Scenarios

1 Ginzburg-Landau equation

Model equation for

"Superconductivity" and "Phase transition"

Time Dependent Ginzburg-Landau Equation

$$\eta\psi_t = \Delta\psi + \kappa^2(1 - |\psi|^2)\psi$$

Stationary Case :

$$0 = \Delta\psi + \kappa^2(1 - |\psi|^2)\psi$$

○ ψ:complex valued function

$|\psi| = 1$::Superconducting state

ψ=0::Normal state

η, κ:positive constants

○ Ginzburg-Landau Energy Functional

$$F(\psi) = ||\nabla\psi||^2 + \frac{\kappa^2}{2}||1 - |\psi|^2||^2$$

GL eq is derived from this Energy Functional

2 Example

2.1 Numerical Simulation

(A kind of) Discrete Variational Method

$$F(\psi) = ||\nabla\psi||^2 + \frac{\kappa^2}{2}||1 - |\psi|^2||^2$$

"Finite Difference scheme" $+$

"(modified) Steepest Descent Method"

[Structure Preserving]

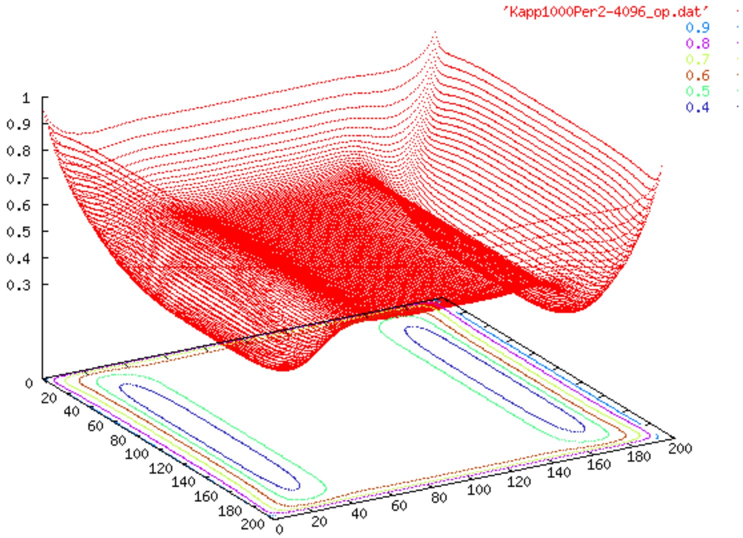
2.2 Emergence of a pair of Vortices

Domain Ω: $[0,1] \times [0,2]$

BC : $\psi(x,0) = \psi(x,2) = e^{(6i\pi x)}$

$+$periodic

IC : $\psi_0(x,y) = e^{(6i\pi x)}$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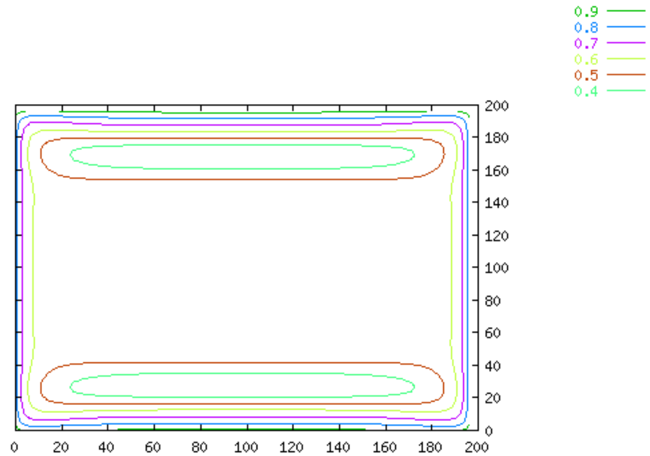
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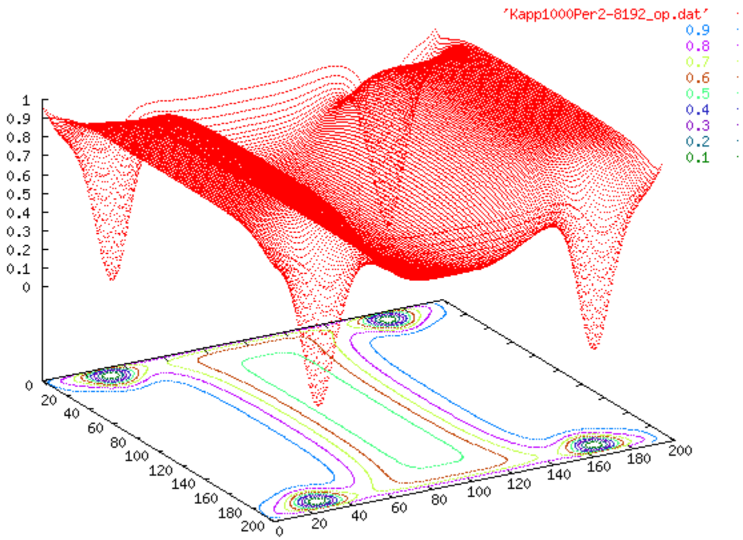
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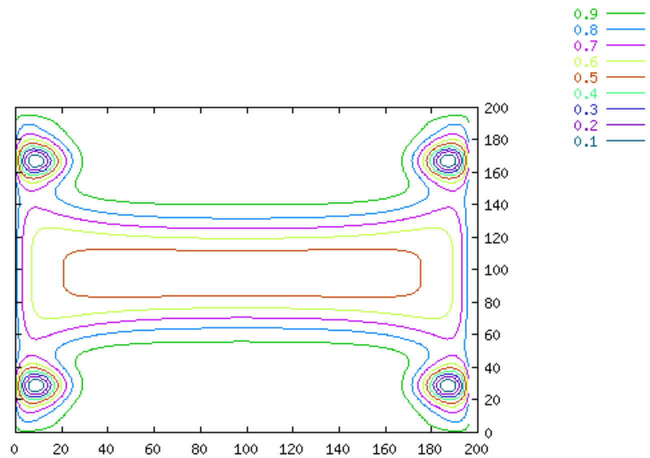
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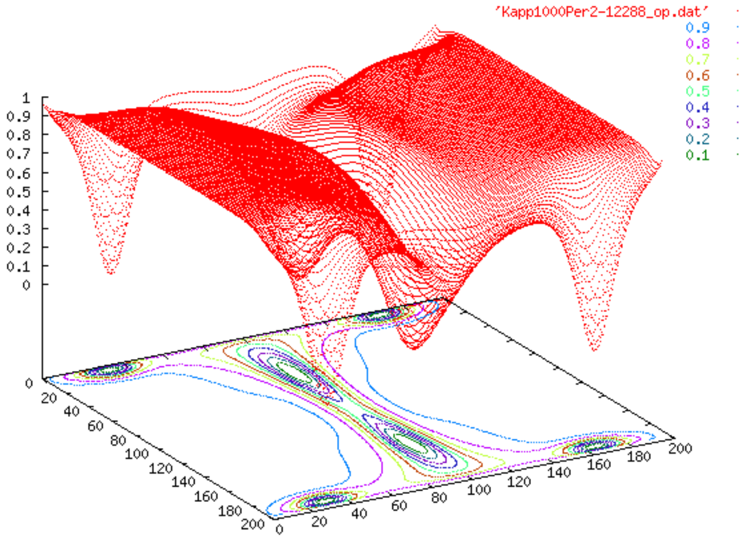
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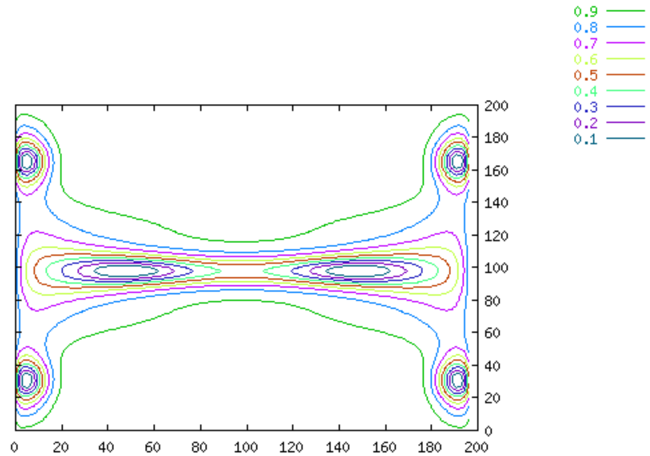
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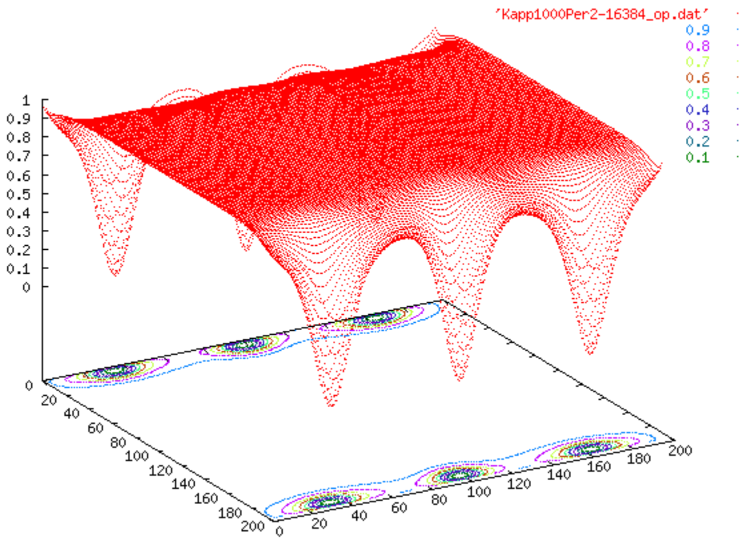
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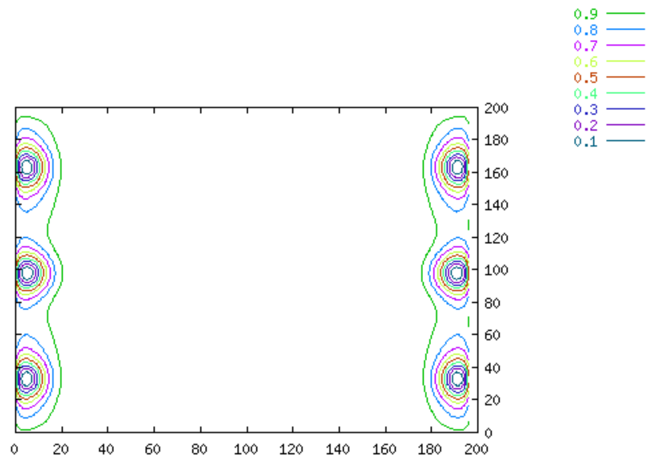
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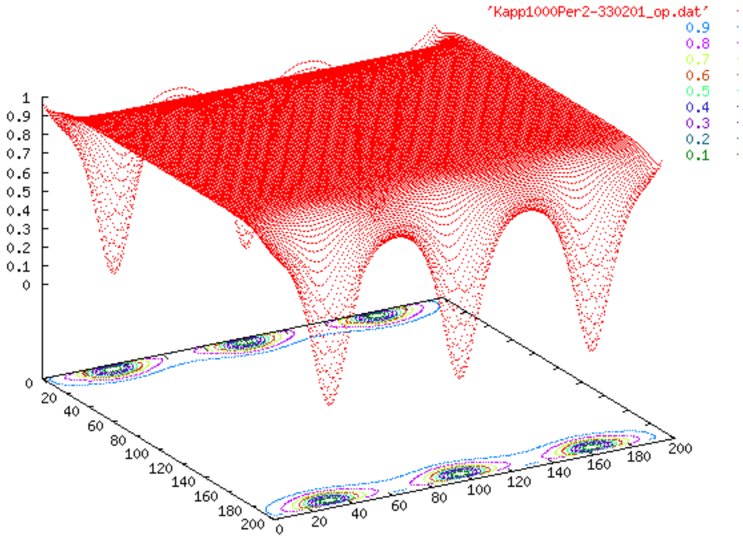
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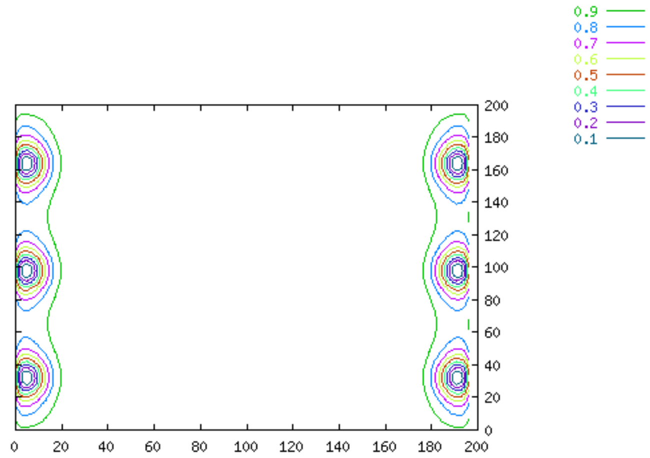
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2.3 Annihilation

Domain : $\Omega_1 \backslash \Omega_2$

$$= [0,1] \times [0,1] \backslash [0.3, 0, 7] \times [0.3, 0.7]$$

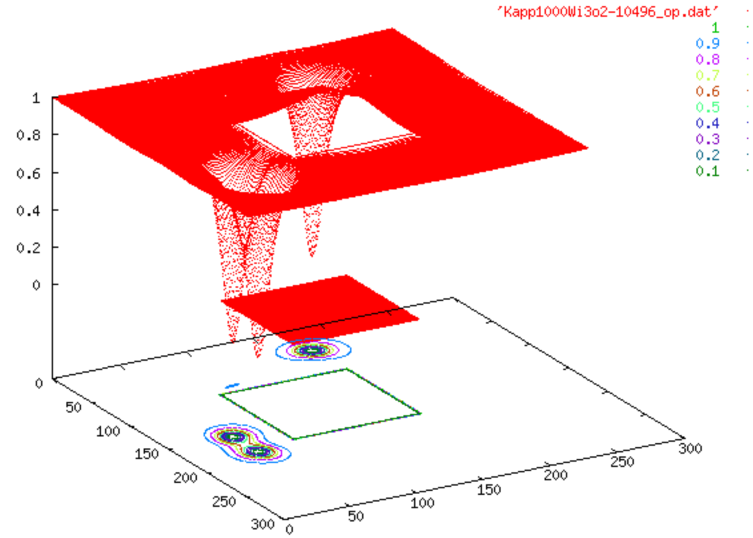
BC: $e^{(2i\theta_1)}$ on $\partial\Omega_2$, $e^{(3i\theta_2)}$ on $\partial\Omega_1$

$\theta_1 = atan((y - 0.49)/(x - 0.51)),$

$\theta_2 = atan((y - 0.51)/(x - 0.49))$

A pair of vortices collide each other

$\longrightarrow$ No vortices.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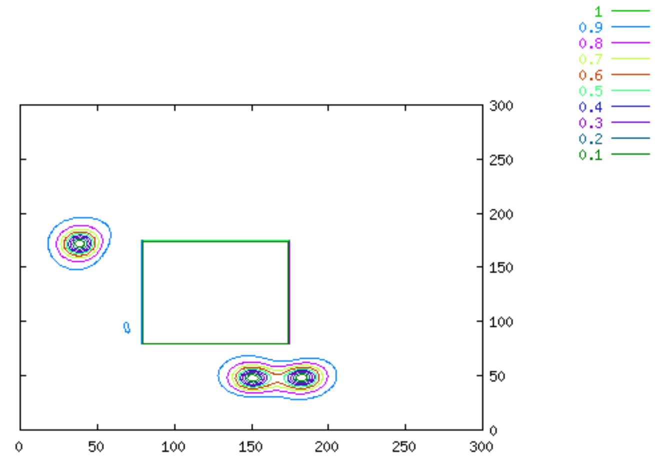
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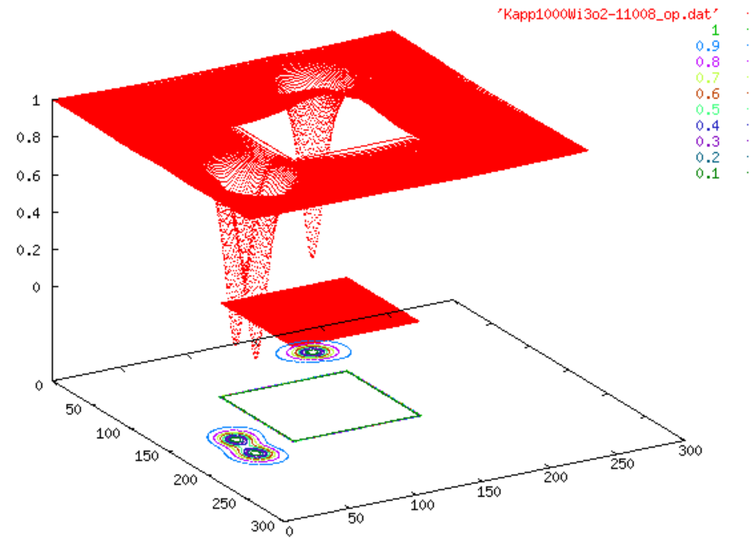
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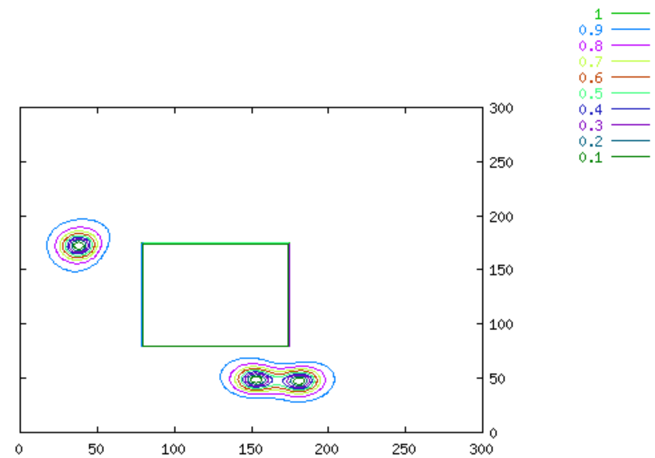
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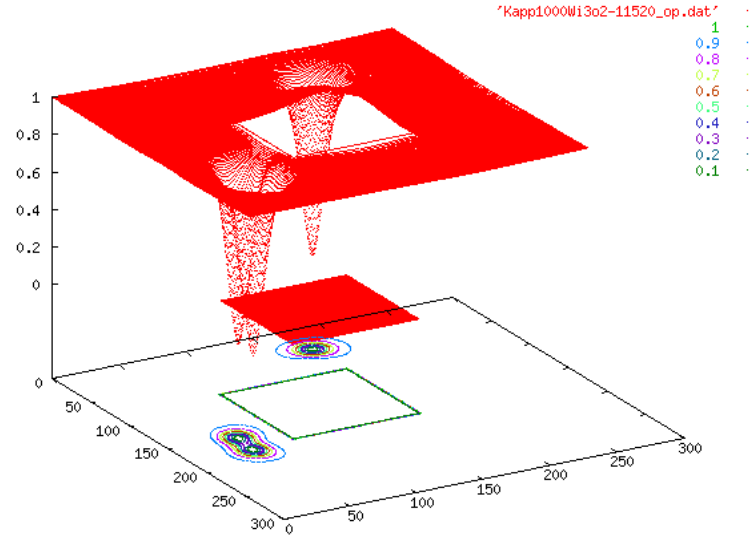
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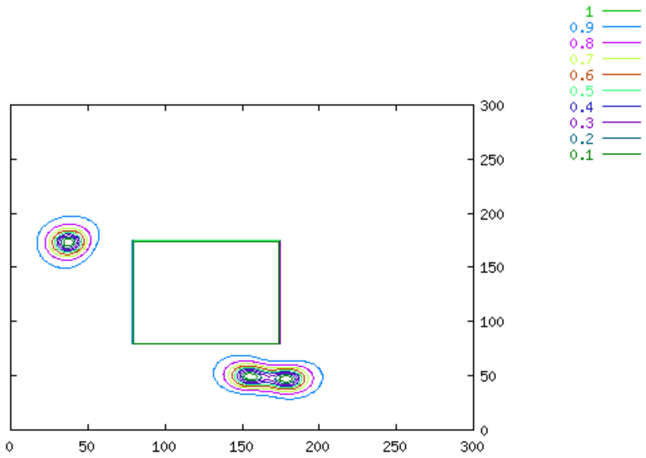
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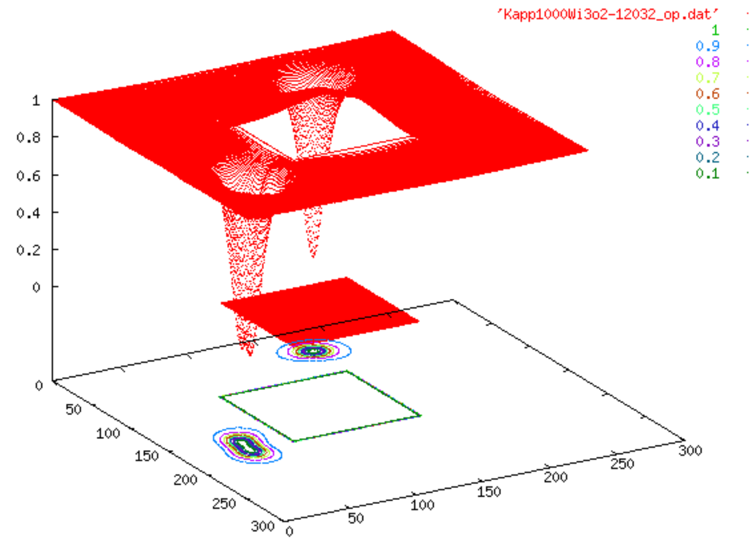
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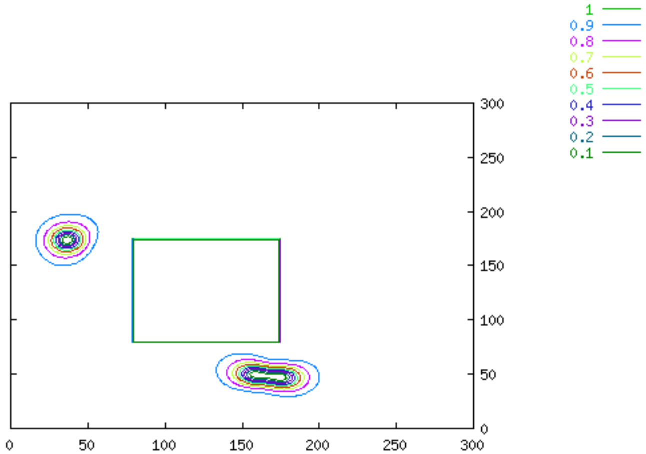
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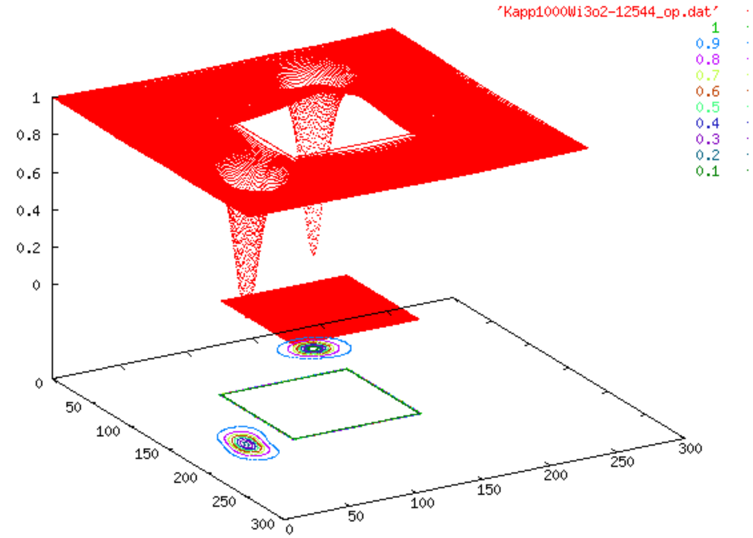
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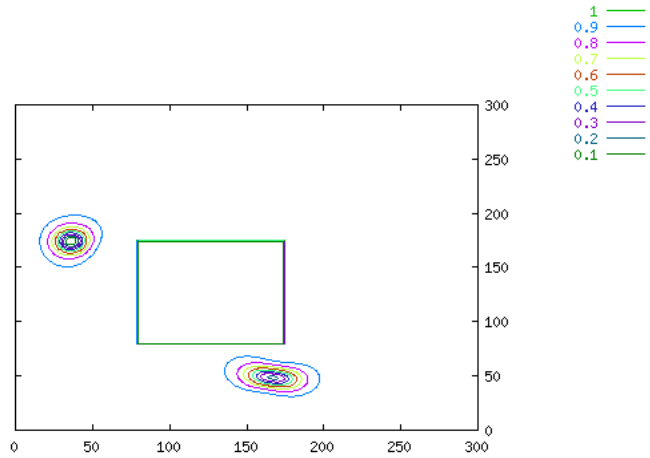
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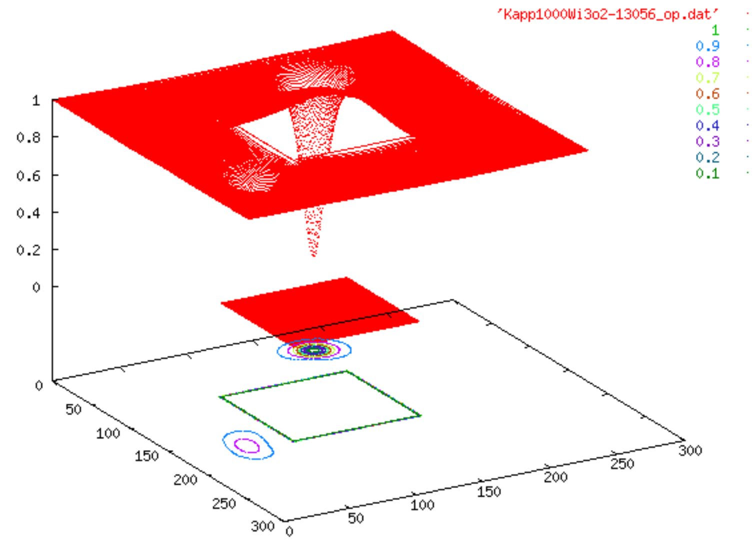
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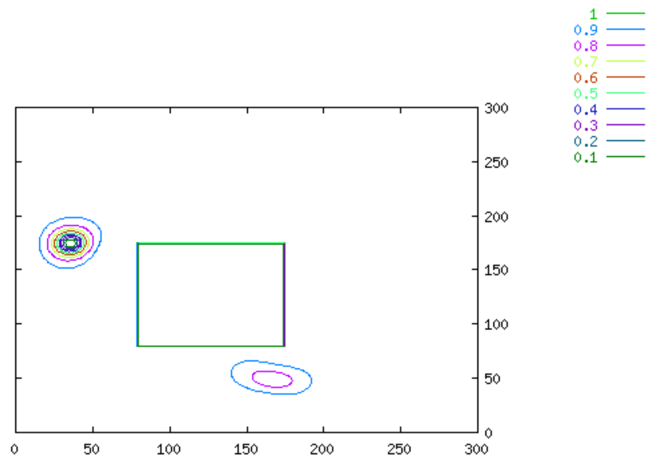
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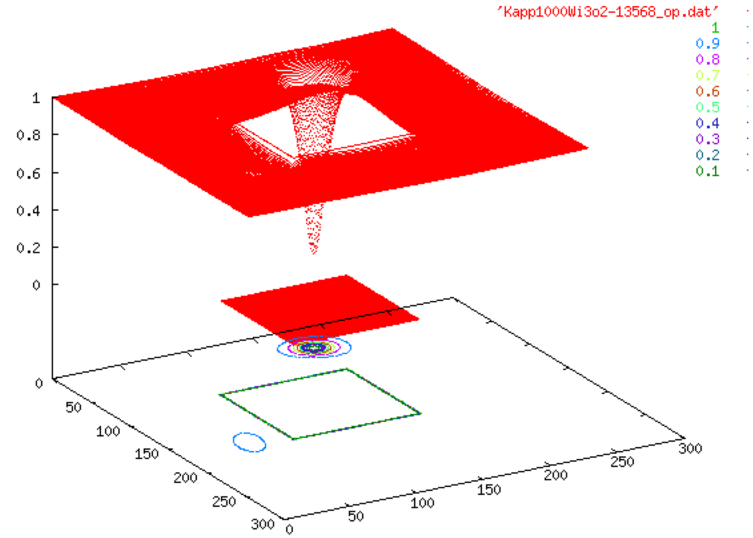
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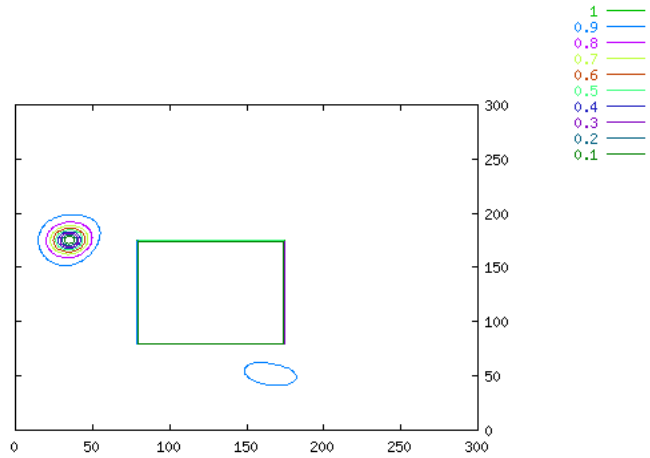
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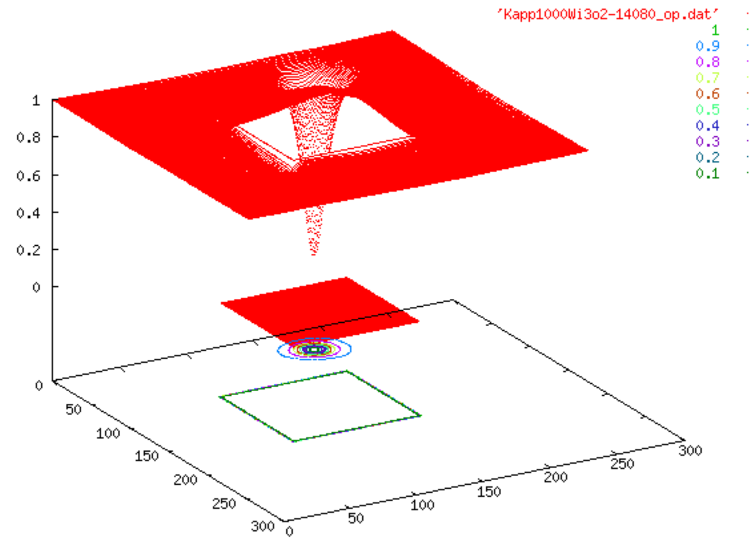
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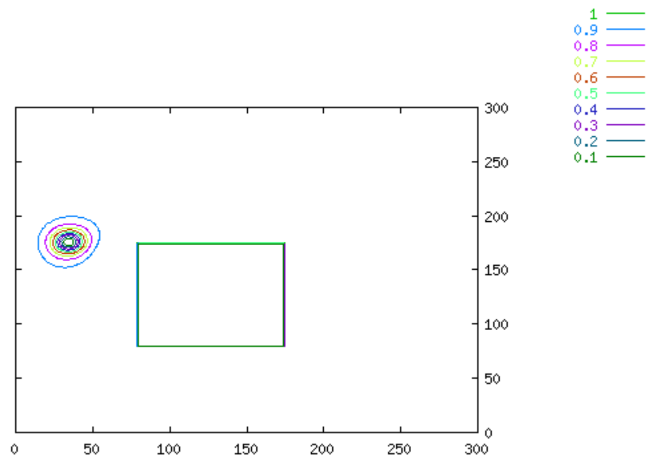
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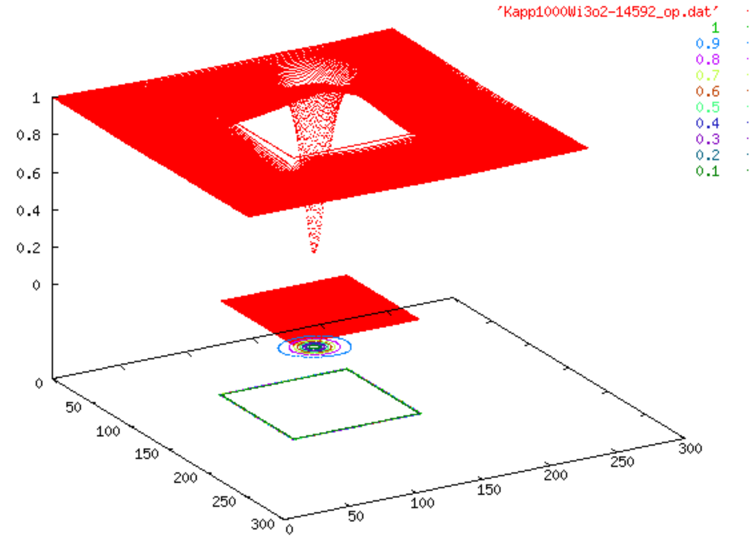

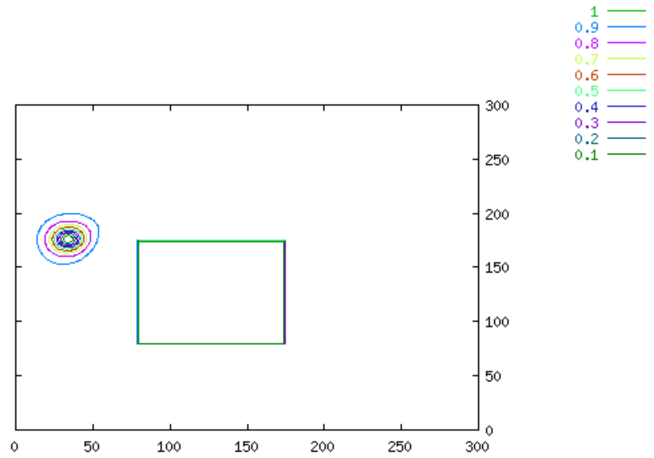

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We want to understand these phenomena.

3 Vortex Solution

Vortex Solution:Solution with zero points

○ Why zero "point" in 2D case?

Take $\psi = u + iv$ $(u, v$:real valued) , then GL equation is....

$$\begin{cases} \Delta u + \kappa^2(1 - |\psi|^2)u = 0 \\ \Delta v + \kappa^2(1 - |\psi|^2)v = 0 \end{cases}$$

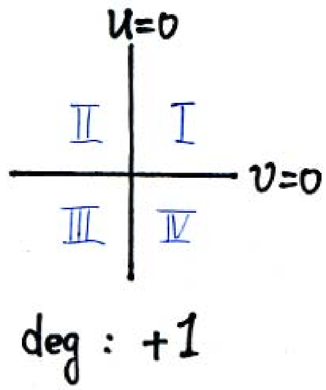
2D case: the set $\{\psi = 0\}$ is intersection of zero level sets $\{u = 0\}, \{v = 0\} \rightarrow$ vortex points

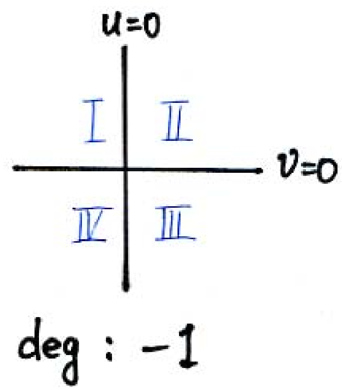
degree of the vortex point

[Definition]

$$deg(g, \partial B_\varepsilon(z_j)) \equiv \frac{1}{2\pi} \oint_\Gamma g \times g_\theta d\theta$$

θ: anticlockwise direction





[What is to say...]

$$\begin{array}{lll} I & : & u > 0, v > 0 \\ III & : & u < 0, v < 0 \end{array} \qquad \begin{array}{lll} II & : & u < 0, v > 0 \\ IV & : & u > 0, v < 0 \end{array}$$

Known results

[Bethuel,Brezis,Helein]

Ω is simply connected and starshaped

$$\Delta\psi + \kappa^2(1 - |\psi|^2)\psi = 0 \ \text{ in } \ \Omega$$

$$\psi = g \ \text{ on } \ \partial\Omega, \ \deg(g, \partial\Omega) = d$$

Then, there are d vortices with $+1$ degree.

○ Breaking Condition "simply connected"

ex.[Berlyand,Mironescu]

Ω, ω : Simply Connected $\quad (\omega \subset \Omega)$

and not thin domain

[H^1-capacity$Cap(\Omega \backslash \omega) < \pi$]

$$\Delta\psi + \kappa^2(1 - |\psi|^2)\psi = 0 \quad \text{in } \Omega\backslash\omega$$

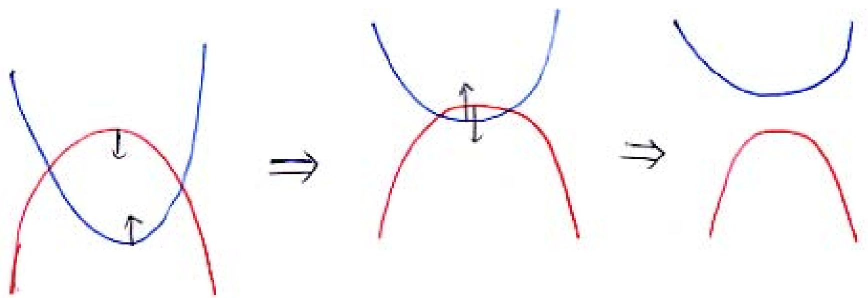
$$\psi = g \ \text{ on } \partial\Omega\backslash\omega, deg(g, \partial\Omega) = 1, \ deg(g, \partial\omega) = 1.$$

There exists a solution with two vortices, one's degree is $+1$, the others degree is -1.

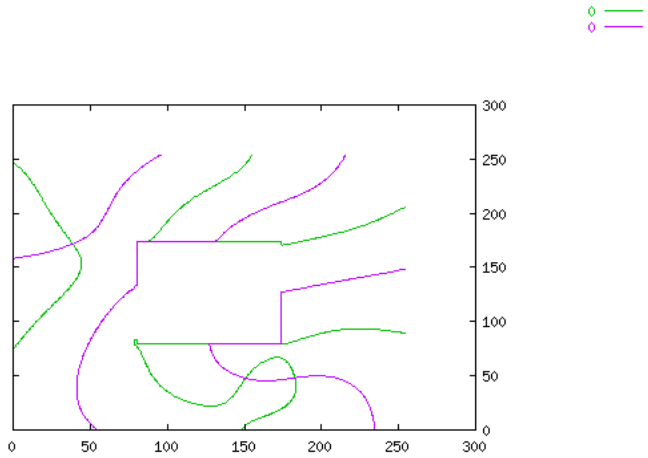
4 Scenarios

4.1: Annihilation of vortices

A pair of vortices which has degree +1 and -1 collide each other $\longrightarrow$ No vortices.



"Disapperance of intersection of two zero level sets"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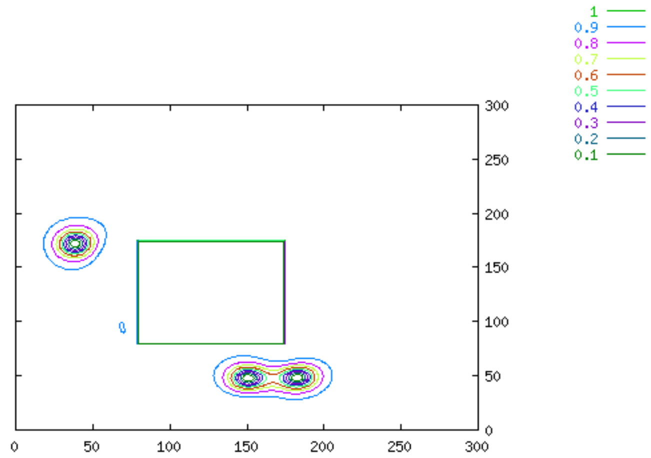
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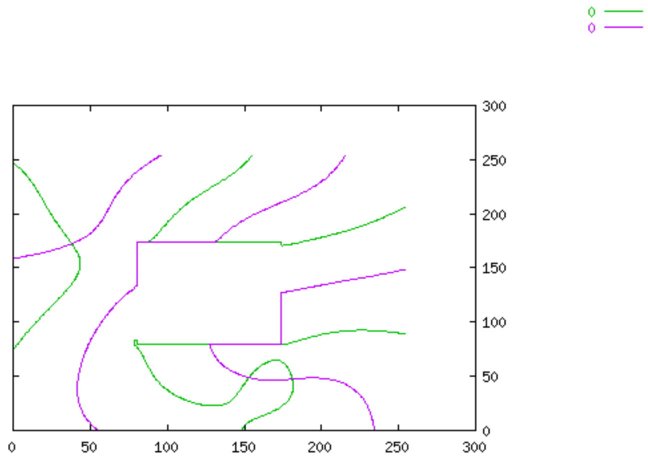
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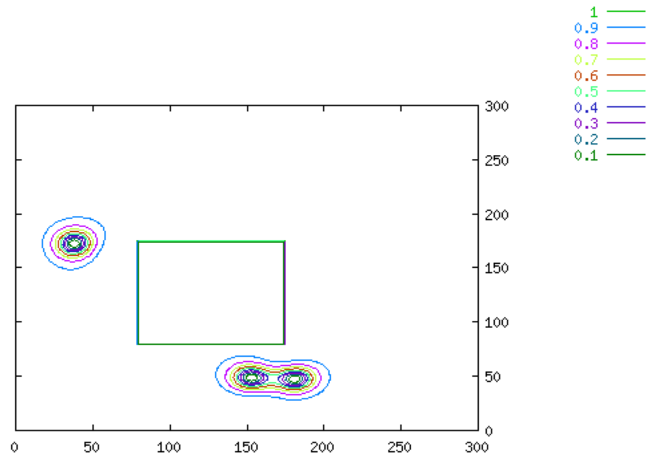
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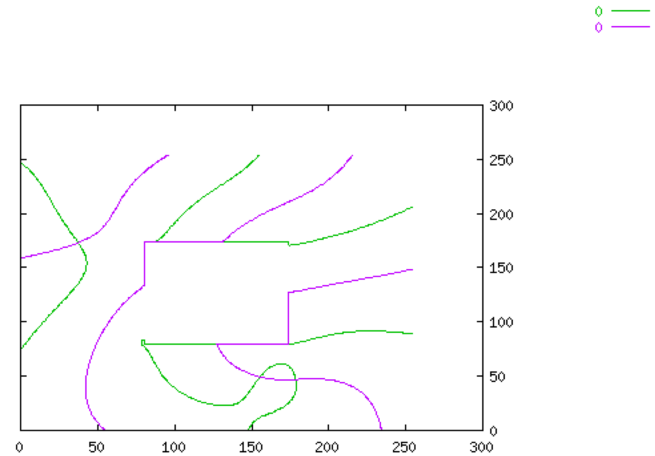
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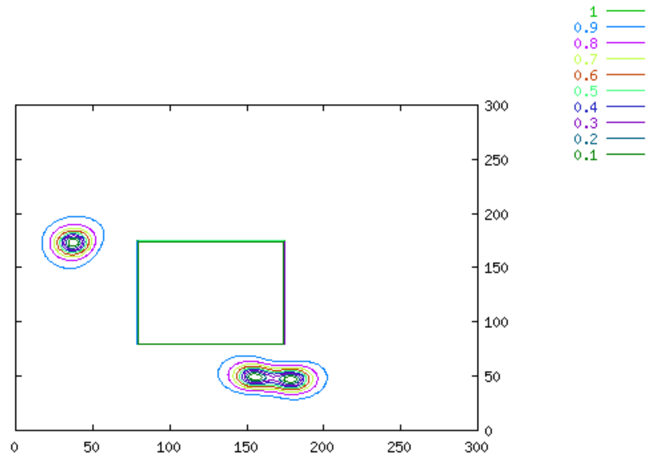
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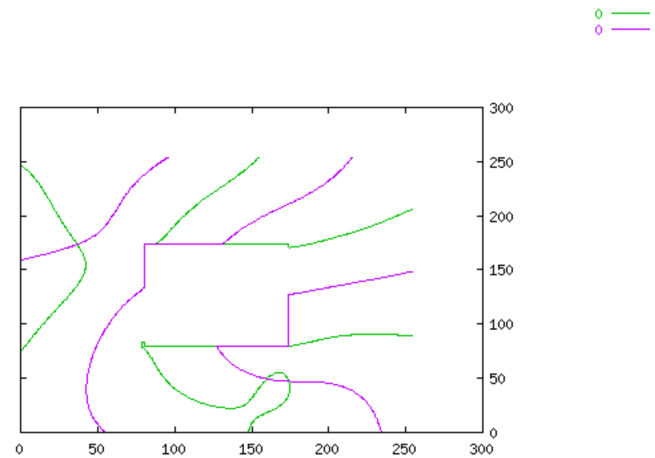
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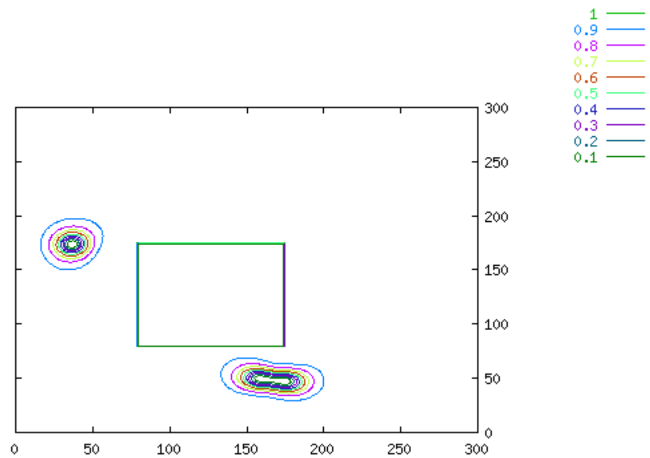
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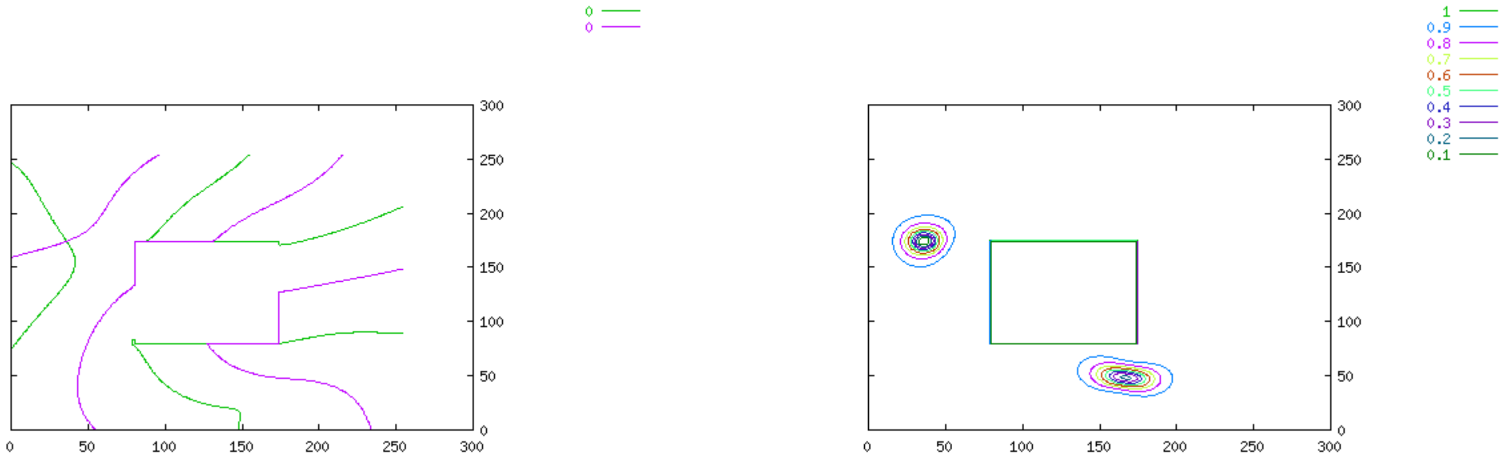
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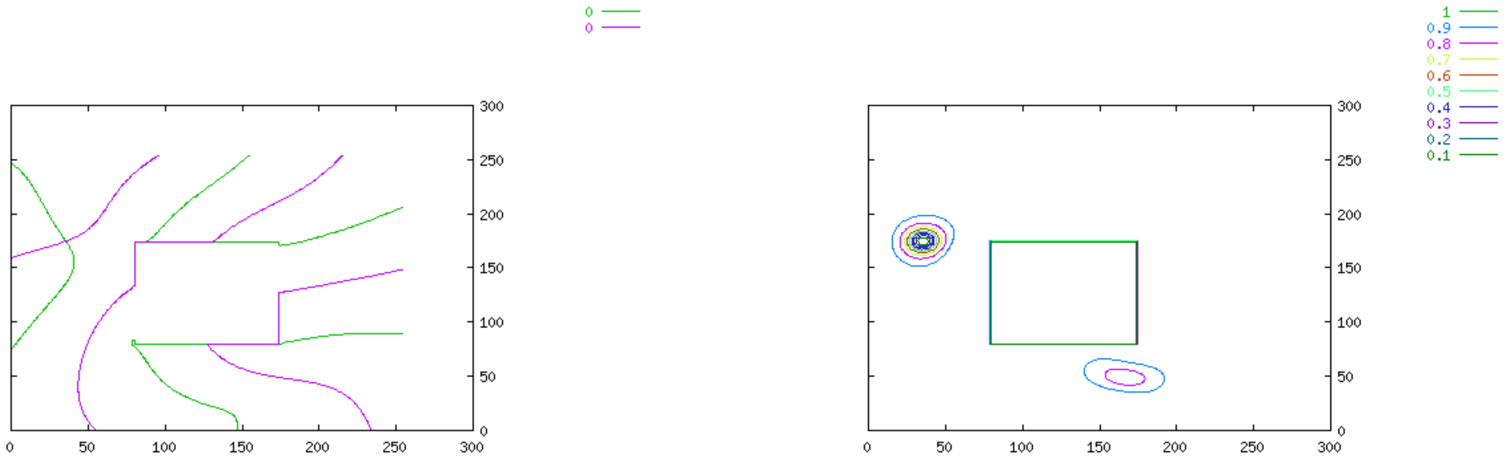
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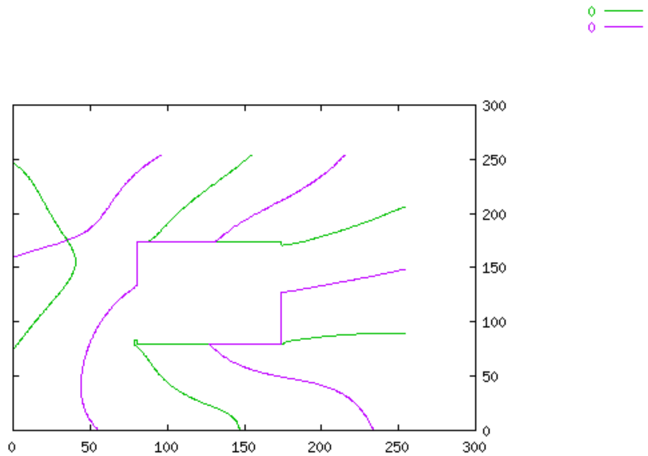
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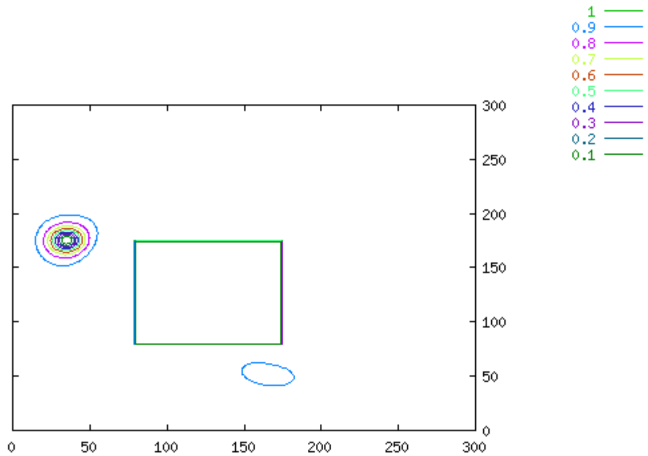
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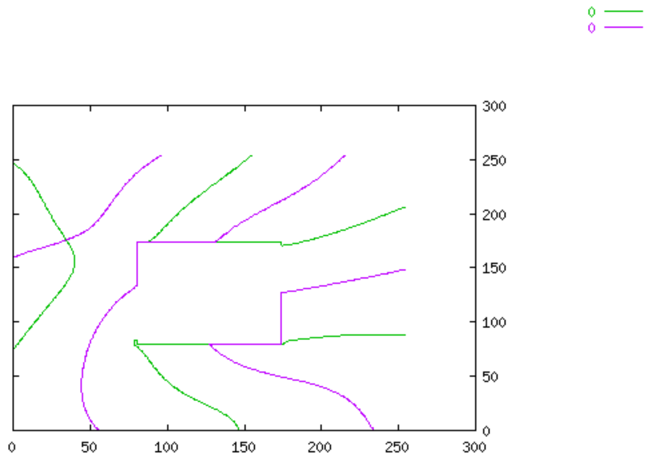
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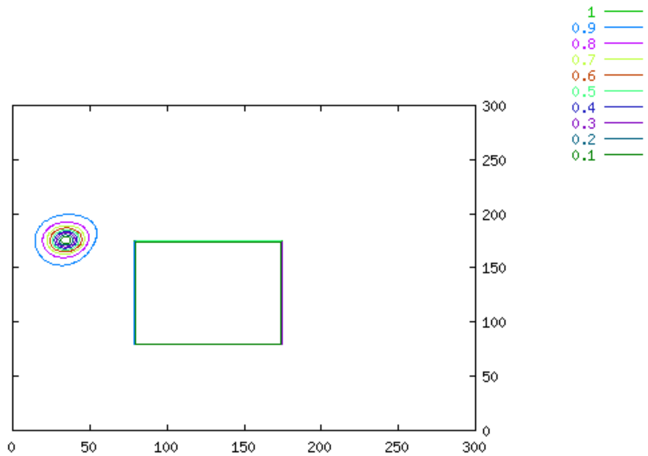
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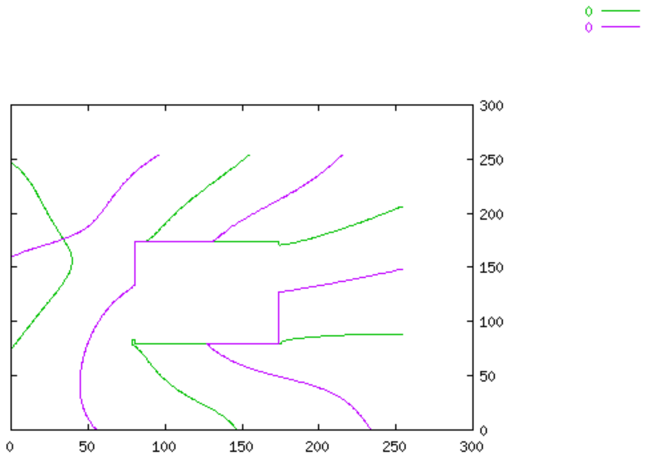
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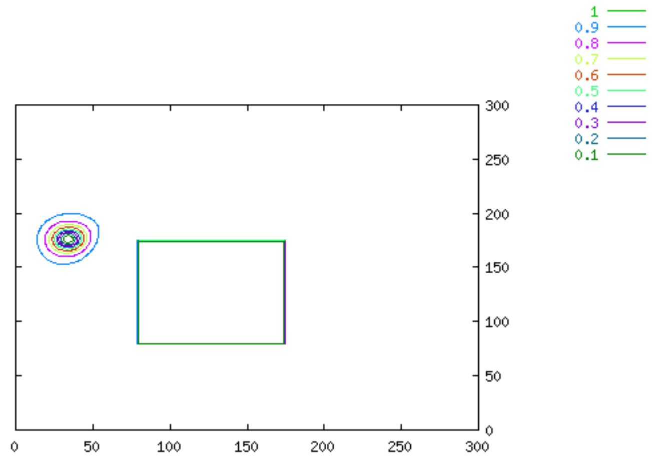
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(End)

[observation]

The behaviour of a zero level set

in 2D scalar heat equation:

The velocity of a point in zero level set $\mathbf{x}(t)$ is

$$\eta\frac{d}{dt}\mathbf{x}\cdot\nabla u = -\Delta u \quad \Longrightarrow \frac{d}{dt}\mathbf{x}\cdot\mathbf{n} = (-H + \frac{\nabla|\nabla u|^2}{|\nabla u|^2}\cdot\mathbf{n})/\eta$$

(H:the curvature, $\mathbf{n}$:the unit normal vector)

If $|\nabla u|^2$is sufficiently large or flat

on $\Gamma_u = \{u = 0\}$, Γ_u behave like curvature flow.

(As long as the area surrounded by Γ_u is not small, $|\nabla u|^2$ is large on Γ_u.)

[Notation]

$A_u(t)$: Domain serrounded by Γ_u,

$A_v(t)$: Domain serrounded by Γ_v

$\bar{\mathbf{x}}$:center of gravity of A_u

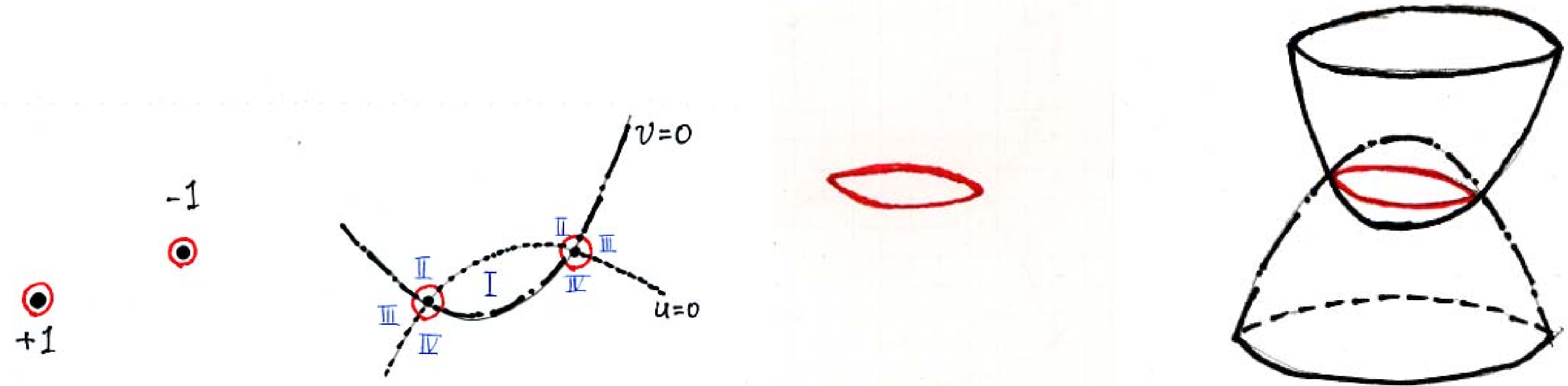
$\bar{\mathbf{y}}$:center of gravity of A_v

(Special Case)

Proposition 1 Let A_u and A_v are convex, large domain.

If $\bar{\mathbf{x}} \notin A_v$ $\bar{\mathbf{y}} \notin A_u$, there exist $T > 0$ s.t. a pair annihilation occur at t=T.

Note: In 3D case, this corresponds to disappearance of vortex ring.



Sketch of the Proof

Since, it is difficult to detect correct zero level set, we prepare the approximated area $|A_\varepsilon|$ and center of gravity $\bar{\mathbf{x}}_\varepsilon$ of A_ε.

$$\chi_\varepsilon \equiv (1 + \frac{u}{\sqrt{u^2+\varepsilon^2}})/2$$

$$|A_\varepsilon(t)| = \int_{\mathbf{R}^2} \chi_\varepsilon dx$$

$$\bar{\mathbf{x}}_\varepsilon(t) = \int_{\mathbf{R}^2} \mathbf{x}\chi_\varepsilon dx / |A_\varepsilon(t)|$$

$\tilde{\mathbf{x}}_\varepsilon(t)$: center of gravity of Γ_u

$$\frac{d}{dt}|A_\varepsilon(t)| = \int_{\mathbf{R}^2} (1-\chi_\varepsilon^2) \frac{u_t}{\sqrt{u^2+\varepsilon^2}} dx < 0$$

$$\frac{d}{dt}\bar{\mathbf{x}}_\varepsilon(t) = \int_{\mathbf{R}^2} (1-\chi_\varepsilon^2) \frac{\mathbf{x}-\bar{\mathbf{x}}_\varepsilon}{|A_\varepsilon|} \frac{u_t}{\sqrt{u^2+\varepsilon^2}} dx$$

$\frac{d}{dt}\bar{\mathbf{x}}_\varepsilon(t)$ is much smaller than $\frac{d}{dt}|A_\varepsilon(t)|$.

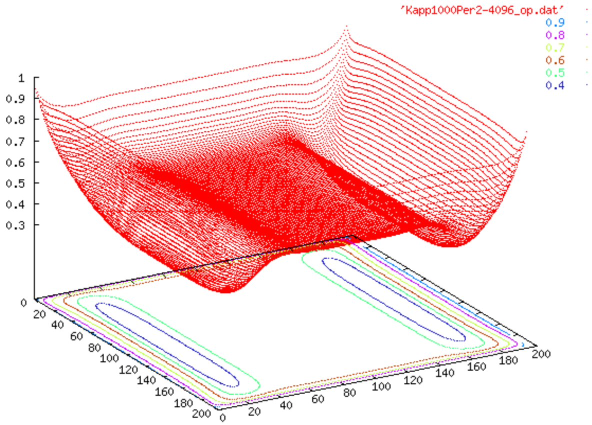
$$\left|\frac{d}{dt}\bar{\mathbf{x}}_\varepsilon(t)\right| \sim \frac{|\tilde{\mathbf{x}}_\varepsilon - \bar{\mathbf{x}}_\varepsilon|}{|A_\varepsilon|} \left|\frac{d}{dt}|A_\varepsilon(t)|\right|$$

So there exist $T > 0$ s.t. $A_u(t) \cap A_v(t) = \emptyset$, $t > T$

4.2 Emergence of Vortices

Since $|\nabla u|$ tend to be large on zero level set, the zero level set(curve) should be short.

$$\Longrightarrow \begin{cases} \text{Shortening like curvature flow} \\ \text{Reconnection} \end{cases}$$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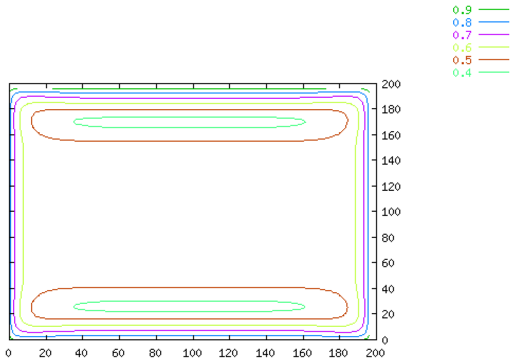
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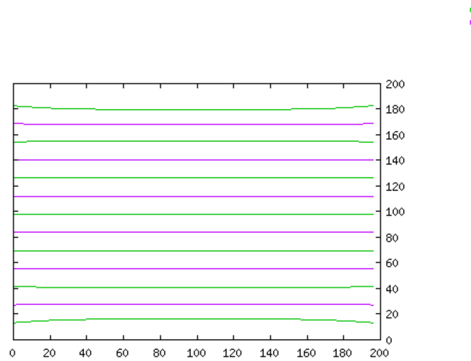

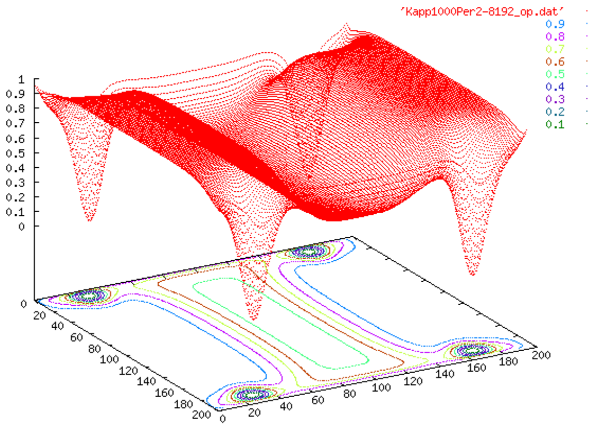
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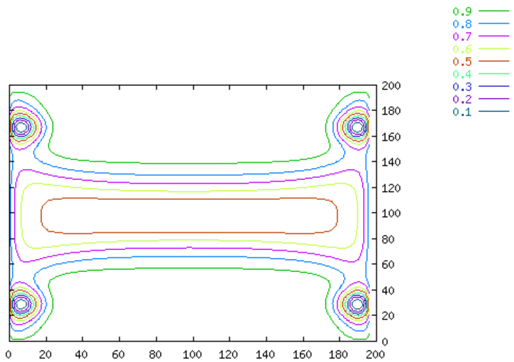
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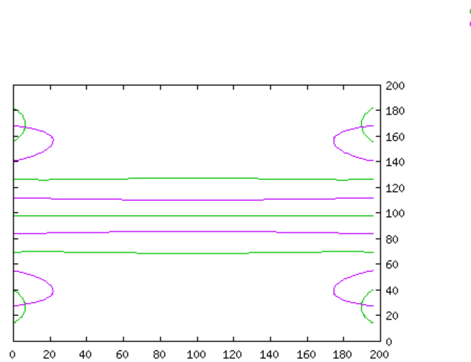

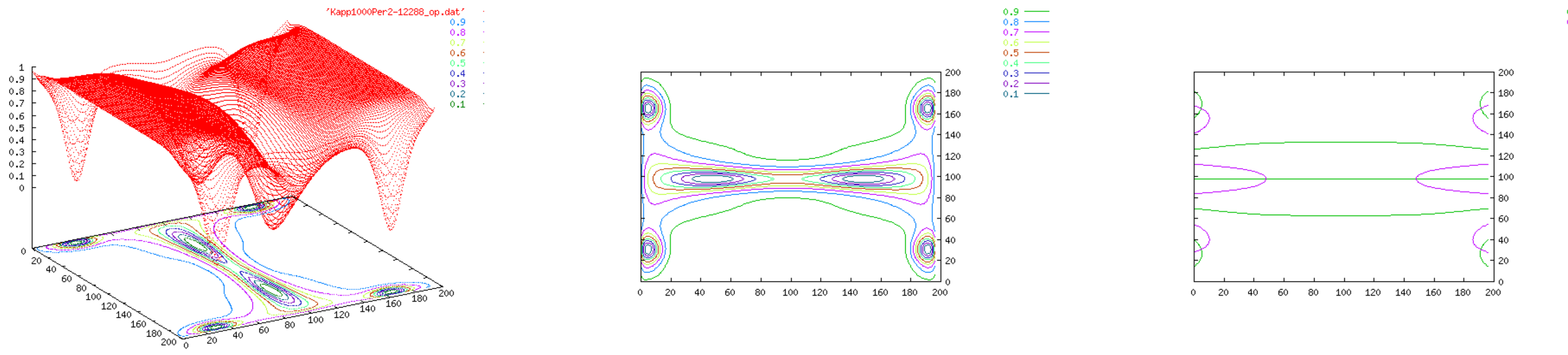
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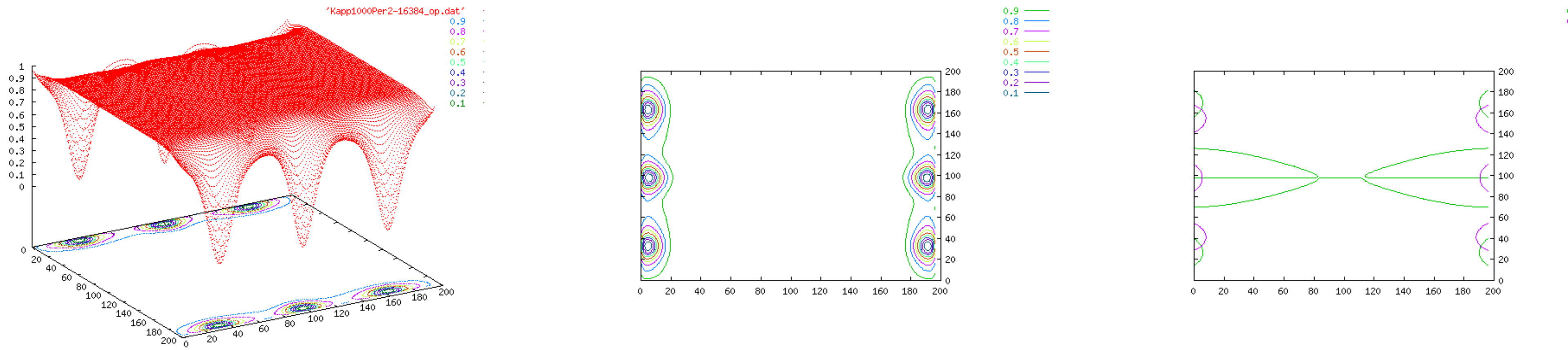
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(End)

Thank you for your attention.