The Motion of Particples Driven by Surface Energy on the Boundary of a Smooth Domain: Peak-like Solutions to Nonlinear Parabolic Equations

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We view particles as peak-like approximate minimizers for the energy functional

$$E(u) \equiv \int_\Omega (\frac{\varepsilon^2}{2}|\nabla u|^2 - F(u)),$$

where $F(u)$ may be like

$$\frac{u^{p+1}}{p+1} - \frac{u^2}{2}, \quad p > 1,$$

or some more general nonlinearity having a similar shape for $u > 0$.

I say, 'approximate' minimizers since I discuss non-equilbrium states, which have 'condensed' to localized states and which evolve by the gradient flow of E.

Thus, particles are peak-like states for the equation:

$$(\dagger) \quad u_t = \varepsilon^2 \Delta u + f(u) \quad (t,x) \in [0,\infty) \times \Omega$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

where $0 < \varepsilon << 1$ and $f = F'$ is such that there is a non-degenerate positive radially symmetric ground state (e.g. $u^p - u$, $p > 1$, subcritical, or $u(u-a)(1-u)$ with $0 < a < \frac{1}{2}$). Ω is a smoothly bounded domain in $\mathbb{R}^n$.

There has been great interest in single and multiple peak steady states for this and related equations:

Lin-Ni-Takagi, Ni-Takagi, J. Wei, Winter, Ward, C. Gui, Rabinowitz, Oh, Alama, Fusco, YY Li, Pacella, Adimurthy, ZQ Wang, Dancer, Du, Yan, Alikakos, J. Shi, X. Chen, DelPino, Kowalczyk, Felmer, F-H Lin, etc.

For existence of stationary states for (†) one may use **mountain-pass methods** (Ni, Takagi, et al) to look for critical points of

$$E(u) \equiv \int_\Omega (\frac{\varepsilon^2}{2}|\nabla u|^2 + \frac{u^2}{2} - \frac{u^{p+1}}{p+1}).$$

Alternatively, consider the quadratic functional

$$Q(v) \equiv \int_\Omega (\frac{\varepsilon^2}{2}|\nabla v|^2 + \frac{v^2}{2})$$

and minimize it on the manifold

$$\mathcal{H} \equiv \{v : \int_\Omega v^{p+1} = 1\}.$$

This constraint introduces a Lagrange multiplier and a scaling of the minimizer, $u = \alpha v$, gives the desired solution. In this approach it is important that the growth rate $p < \frac{n+2}{n-2}$. We assume this throughout.

IDEAS

The energy minimizing state is localized: To minimize

$$\int_\Omega \frac{v^2}{2}$$

on the manifold $\mathcal{H}$ we see that $v \equiv |B_r|^{-\frac{1}{p+1}}$ gives a value of $\frac{1}{2}|B_r|^{1-\frac{2}{p+1}}$, and since $p > 1$, this approaches 0 as $r \to 0$.

The shape of the set S on which v takes a positive constant value is immaterial, only its measure is important. However, by including the gradient term in Q, the approximate minimizers are no longer piecewise constant, but have a steep slope around the boundary of S. Also, since the gradient term contributes to the value of Q in an amount proportional to the $(n-1)$-dimensional measure of the boundary of S, it is reduced by making the boundary spherical. Furthermore, the radius is now prevented from reducing to 0, since that would cause the gradient term to become unbounded.

The minimizing shape is called the *ground state.*

We do even better by using the boundary of Ω as part of the boundary of the set, eliminating the need for a steep gradient there.

This suggests that minimizing Q over $\mathcal{H}$, when ε is small, results in a state having a sharp peak concentrated on the boundary of the domain Ω.

Finally, placing the peak at the point on $\partial\Omega$ where the mean curvature is greatest, one can use more of $\partial\Omega$ to enclose the peak, using less interfacial (gradient) energy. These are the heuristics behind the detailed and rigorous analysis in the papers by Ni and Takagi and lead us to other approaches and results.

The Dynamical Systems Approach: Find a manifold of approximate solutions, $\mathcal{M}$, near a rescaled ground state on $\mathbb{R}^n$, and then seek a better aproximation to an invariant manifold $\bar{\mathcal{M}}$ as a graph over $\mathcal{M}$, that contains true stationary solutions as critical points of the induced flow on that manifold.

For example, in the dynamical systems approach, let

$$\mathcal{M} = \{u(\cdot, \zeta) : \zeta \in D\}, \quad (D \subset \mathbb{R}^k \quad \text{parameter set})$$

be the approximate manifold (known) and search for

$$\bar{\mathcal{M}} = \{u(\cdot, \zeta) + v(\cdot, \zeta) : \zeta \in D\}$$

such that

$$c(\zeta) \cdot \mathrm{grad}_\zeta(u + v) = \varepsilon^2 \Delta(u + v) + f(u + v),$$

'vector field' given by RHS of the PDE is tangent to the manifold.

Find functions v and c over $\mathcal{M}$, c giving the velocity field on $\bar{\mathcal{M}}$.

In seeking stationary states, consider instead

$$c(\zeta) \cdot \mathrm{grad}_\zeta(u) = \varepsilon^2 \Delta(u + v) + f(u + v).$$

With G.Fusco we followed this approach for the Cahn-Hilliard equation.

Use the idea that the peak states are
strongly unstable in certain directions,
strongly stable in a finite co-dimensional submanifold,
very weakly stable or unstable in "translational" directions (along $\bar{\mathcal{M}}$).

I am interested in finding a global truly invariant manifold of spike-like solutions to the parabolic equation that exist globally and that contains the stationary solutions.

Abstract Theory: Infinite-dimensional dynamical systems

- **Invariant Manifolds**
  Let X be a Banach space.

$T \in C^k(X, X)$, $k \geq 1$.

$M \subset X$: a C^k submanifold

- M is inflowing invariant if $T(M) \subset M$ and

$$d(T(M), \partial M) > 0$$

.

- M is overflowing invariant if $\exists M_1 \subset M$ such that $T(M_1) = M$ and $d(M_1, \partial M) > 0$.

Examples: Fixed points, invariant tori, and their stable and unstable manifold, inertial manifolds, etc.

• **Background:**

The beginings:

Poincare; Hadamard; Perron; Lyapunov, etc.

Initial blossoming:

Krylov, Bogoliubov and Mitropolski; Hale; Kyner; Levinson; Diliberto; Hufford; Jarnik and Kurzweil; Marcus; McCarthy; Sternberg; etc.

Maturity:

Fenichel; Hirsch, Pugh and Shub; Mañé; Chow, Liu, and Yi; and others.

◦ Infinite Dimensional Dynamical Systems

Henry; Ball; Ruelle; B. and Jones; Chow and Lu, B., Lu, and Zeng; Sell, Mallet-Paret, Pliss, etc.

• **How to find invariant manifolds?**

• Direct method: using conserved quantities and symmetries.

Good for Hamiltonian systems

• Perturbation method:

1. Local stable, unstable, center-stable, center-unstable manifolds near fixed points, periodic orbits, etc.
2. Persistent (large) invariant manifolds for perturbed systems;

• **Conditions for unique persistence of smooth invariant manifolds?**

For manifolds without boundary: Normal Hyperbolicity

For inflowing manifolds: unstably normally hyperbolic

For overflowing manifolds: stably normal hyperbolic

Approximately invariant manifolds.

• Under what conditions, does there exist a true invariant manifold near an approximately invariant manifold?

• Is there a concept of normal hyperbolicity for *approximately* invariant manifolds?

• M is **approximately inflowing invariant** under T if there is a map $T_0 \in C^0(M, M)$ and $\eta, r > 0$ such that

$$|T(m) - T_0(m)| \leq \eta, \quad d(T_0(M), \partial M) \geq r.$$

• M is **approximately unstably normally hyperbolic** if at any $m \in M$, there is a splitting

$$X = X_m^u \oplus X_m^c$$

such that

1. X_m^u and X_m^c are Lipschitz in m and the 'angle' between them is uniformly bounded below.
2. X_m^c is approximately $T_m M$ and is approximately invariant under DT
3. $\Pi^u DT|_{X^u}$ isomorphically expands and does so to a greater degree than does $DT|_{X^c}$, where Π^u is the projection to X^u.

- Parameters:
  - σ: Local error between M and X_m^c and the error between $X_{T(m)}^c$ and $DT(X_m^c)$;
  - η: The error between $T(M)$ and M;
  - λ: Measurement of expansion in X^u;
  - B: Measurement of the lower bound of the 'angle' between X^c and X^u;
  - r: radius of the working neighborhood around M;
  - B_1: Upper bound of D^jT, $1 \leq j \leq k$, in the r-neighborhood of M;
  - L: Lipschitz constant of $X_m^{u,c}$ in m;

Theorem 1 (**BLZ**) *For $k > 1$ (or $k = 1$ and M is precompact) there exist*

$$0 < \sigma_0 = \sigma_0(B, B_1, \lambda), \textit{ and } 0 < \delta, \eta_0,$$

depending on $B,\ B_1,\ \lambda,\ r,\ L$, such that if $\sigma \in (0, \sigma_0)$, $\eta \in (0, \eta_0)$, then there exists a C^k inflowing invariant manifold $\tilde{M}$ in a δ-neighborhood of M.

• A similar theorem holds for Approximately Stably Normally Hyperbolic *Overflowing* manifolds.

• Approximate normally hyperbolic invariant manifolds

• M is approximately normally hyperbolic if at any $m \in M$, there is a splitting

$$X = X_m^s \oplus X_m^u \oplus X_m^c$$

such that

1. $X_m^{s,u,c}$ are Lipschitz in m and the 'angles' between them are uniformly bounded below.
2. X_m^c is approximately $T_m M$ and X^s and X^c are approximately invariant under DT
3. $\Pi^u DT|_{X^u}$ isomorphically expands and does so to a greater degree than does $DT|_{X^c}$ while $\Pi^s DT|_{X^s}$ contracts and does so to a greater degree than does $DT|_{X^c}$.

Remarks

• If M is approximately inflowing invariant, then $M^s = \{m + x^s : m \in M, x^s \in X_m^s,\ |x^s| < \delta\}$ is approximately unstably normally hyperbolic inflowing invariant.
$\implies$ $\exists$ invariant stable mnfld W^s diffeomorphic to M^s.

• Existence and non-uniqueness of an inflowing invariant manifold diffeomorphic to M.

Some ideas about the proofs:

Consider the more difficult case where

$$M^s = \{m + x^s : m \in M, x^s \in X^s_m,\ |x^s| < \delta\}$$

is approximately unstably normally hyperbolic inflowing invariant.

Let Γ be the set of μ-Lipschitz graphs over M^s for some small μ.

If we were considering the approximately stably normally hyperbolic overflowing case, M^u, then we would map each member of Γ forward and as the graphs are "stretched tangentially" and "compressed normally", they are mapped by T into Γ and the mapping is a contraction in the sup norm.

In the case of M^s one would like to take the inverse image of each member of Γ under T. However, T is not a homomorphism (e.g. the time-1 map of a parabolic flow), and so it seems that one cannot find a preimage of the graph.

What we show is the following (omitting the technical assumptions of approximately unstably normally hyperbolic inflowing invariant)

Lemma Let $h \in \Gamma$. For each point $m + x^s \in M^s$ there is a point $x^u \in X^u_m$ such that $T(m+x^s+x^u) \in \text{graph}(h)$.

This is proved using a contraction argument but the essence is that the unstable fiber is stretched by T and so the image intersects $\text{gr}(h)$.

This provides a graph $\tilde{h}$ over M^s that maps *into* h. Using the approximate normal hyperbolicity, we show that $\tilde{h} \in \Gamma$ and that the mapping $h \to \tilde{h}$ is a contraction, and has a fixed point h_0.

We call the graph of this W^{cs}, the center-stable manifold.

• The proof is complicated by the fact that the vector bundle of stable and unstable subspaces based on M is not trivial and that local coordinate charts must be used to represent graphs and to define what it means to be Lipschitz. The size of the neighborhood in which a coordinate system can be used must be uniformly controlled, so the manifold should not twist too much.

The application

$$(*) \quad u_t = \varepsilon^2 \Delta u + f(u) \quad (t,x) \in [0,\infty) \times \Omega$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

We build an approximately invariant normally hyperbolic manifold by taking the rescaled radially symmetric ground state: w satisfying

$$\begin{cases} \Delta w + f(w) = 0, & y \in \mathbb{R}^n, \\ w(0) = \max w(y), & w > 0, \\ w(y) \to 0, & |y| \to \infty. \end{cases}$$

With $L_0 \equiv \Delta + f'(w) : W^{2,q}(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n)$,

$\sigma(L_0) \cap (-b, \infty) = \{\lambda_1, 0\}$, for some $b > 0$;

$\lambda_1 > 0$ is the principle eigenvalue, and the eigenspace of 0 is spanned by

$$\{\frac{\partial w}{\partial y_j} : j = 1, 2, \ldots, n\}.$$

Define

$$|u|^q_{k,\varepsilon} = \Sigma^k_{i=0} \varepsilon^{qi-n} \Sigma_{|\alpha|=i} |\partial^\alpha u|^q_{L^q(\Omega)}.$$

The phase space will be taken as $X = (W^{2,q}(\Omega), |\cdot|_{0,\varepsilon})$.

For any $p \in \partial\Omega$, let

$$\tilde{w}_{\varepsilon,p}(x) = w(\frac{x-p}{\varepsilon}).$$

Since $\tilde{w}_{\varepsilon,p}$ does not satisfy the boundary condition, it will be modified:

Given any $v : \partial\Omega \to R$, let h be the solution of

$$\begin{cases} \varepsilon^2 \Delta h + f'(0)h = 0, & x \in \Omega, \\ \frac{\partial h}{\partial n} = v, & x \in \partial\Omega. \end{cases}$$

Define a linear operator Bc by $Bc(v) = h$. For $p \in \partial\Omega$, let

$$W_{\varepsilon,p} = \tilde{w}_{\varepsilon,p} - Bc(\frac{\partial \tilde{w}_{\varepsilon,p}}{\partial n}).$$

Define the smooth imbedding $\psi_\varepsilon : \partial\Omega \to L^2(\Omega)$ by

$$\psi_\varepsilon(p) \equiv W_{\varepsilon,p}$$

and the approximate invariant manifold

$$M_\varepsilon = \psi_\varepsilon(\partial\Omega).$$

The boundary correction $Bc(\frac{\partial w_{\varepsilon,p}}{\partial n})$ is (better than) order $O(\varepsilon)$ in terms of $|\cdot|_{k,\varepsilon}$ for any $k \geq 0$.

Let $v(x) > 0$ be the first eigenfunction, corresponding to the eigenvalue λ_1, of the linearized operator L_0.
For any $p \in \partial\Omega$, define

$$\tilde{v}_{\varepsilon,p}(x) = v(\frac{x-p}{\varepsilon}), \quad V_\varepsilon(p) = \tilde{v}_{\varepsilon,p} - Bc(\frac{\partial}{\partial n}\tilde{v}_{\varepsilon,p}),$$

and

$$X^u_{\varepsilon,p} = \text{span}\{V_\varepsilon\},\ X^c_{\varepsilon,p} = T_{\psi_\varepsilon(p)}M_\varepsilon, \quad X^s_{k,\varepsilon,p} = (X^c_{\varepsilon,p} \oplus X^u_{\varepsilon,p})^\perp$$

M_ε is approximately invariant and normally hyperbolic in the sense of the abstract results, provided ε is sufficiently small.

Obtain an inflowing center-stable invariant manifold W^{cs} and an overflowing center-unstable invariant manifold W^{cu}. These are C^j sections of the vector bundles $(M_\varepsilon, X^s_{\varepsilon,k,p})$ and $(M_\varepsilon, X^u_{\varepsilon,p})$, respectively.

By taking their intersection, we obtain an invariant manifold $\tilde{M}_\varepsilon$ in a small $W^{k,q}$ neighborhood of M_ε, which therefore consists of spike-like functions.

Finally, one can compute the vector field on $\tilde{M}_\varepsilon$ induced by the equation, obtaining a dynamical system on $\partial\Omega$ for the location of the peak of the spike.

The results can be summarized as

Theorem

Under the assumptions mentioned above, for any sufficiently small $\varepsilon > 0$, there exists a smooth mapping $\Psi_\varepsilon : \partial\Omega \to W^{2,2}_\varepsilon(\Omega)$ such that

1. For any $q \in (n, \infty)$, there exists $C > 0$ independent of $p \in \partial\Omega$ and $\varepsilon > 0$ such that
$$|\Psi_\varepsilon(p) - w(\frac{\cdot - p}{\varepsilon})|_{C^0((\partial\Omega, \frac{1}{\varepsilon^2}<\cdot,\cdot>), W^{2,2}_\varepsilon(\Omega)\cap W^{2,q}_\varepsilon(\Omega))} \leq C\varepsilon$$
$$|\Psi_\varepsilon(p) - w(\frac{\cdot - p}{\varepsilon})|_{C^1((\partial\Omega, \frac{1}{\varepsilon^2}<\cdot,\cdot>), W^{2,2}_\varepsilon(\Omega)\cap W^{2,q}_\varepsilon(\Omega))} \to 0.$$

2. There exists a unique $\tilde{p} \in \partial\Omega$ such that $\max_{x\in\bar{\Omega}} \Psi_\varepsilon(p)(x) = \Psi_\varepsilon(p)(\tilde{p})$. Moreover $|p - \tilde{p}| < C\varepsilon^2$ for some $C > 0$ independent of $0 < \varepsilon << 1$.

3. $M^*_\varepsilon \equiv \Psi_\varepsilon(\partial\Omega)$ is a normally hyperbolic invariant manifold of the flow generated by the PDE (*).

4. Equation (*) induces a vector field $Y_\varepsilon(p)$ on $\partial\Omega$ that satisfies
$$|Y_\varepsilon(p) - \gamma\varepsilon^3\nabla\kappa(p)| \leq C\varepsilon^4$$
for some $C > 0$ independent of $p \in T_p\partial\Omega$ where $\kappa(p) = H(p)\cdot N(p)$ and $H(p)$ is the mean curvature vector of $\partial\Omega$ and
$$\gamma = \frac{1}{3}\int_{\partial\mathbb{R}^n_+}\left[\frac{w'(|y|)}{|y|}\right]^2 y_j^4 dy > 0.$$